CALCULUS

WORKS OF H. B. PHILLIPS, PH.D.

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BY

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PREFACE

In this text on calculus I have endeavored to emphasize those principles that are found most useful in applications to science and engineering. In the arrangement of material I have merely brought together topics naturally associated in the problems. In the treatment of these topics detailed methods are presented rather as suggestions than as embodying rules necessarily to be followed. The formal work on integration has been segregated so that a choice may be made as to the amount of time assigned to that part. For the convenience of those who do only a limited amount of this formal integration, a table of integrals, including all the forms needed in the problems, has been appended.

The last chapter on differential equations contains a brief treatment of the types most frequently encountered. For a more complete treatment reference is made to my book on Differential Equations.

H. B. PHILLIPS

CAMBRIDGE, MASS., *February*, 1927

CONTENTS

CALCULUS

CHAPTER I

INTRODUCTION

1. Definition of Function. — In many problems we have two variables one of which is expressible in terms of the other. When a value is assigned to one, a value of the other can be calculated. This relation of two variables is expressed by the word function. That is, *a quantity y is called a function of a quantity x if values of y are determined by values of x.*

Thus, the distance moved by a body is a function of the time; for, the time being given, the distance is determined. The area of a circle is a function of its radius; for, the radius being given, the area can be calculated.

It is not necessary that a single value of the function correspond to each value of the variable. Several values may be determined. Thus, if x and y satisfy the equation

$$x^2 - 2\,xy + y^2 = x$$

then y is a function of x. To each value of x correspond two values

$$y = x \pm \sqrt{x}$$

found by solving the equation for y. In an actual problem however one of these might not be used so that y (as used) might be single valued.

A quantity u is called a function of several variables if u is determined when values are assigned to all those variables.

Thus, if $z = x^2 + y^2$, then z is a function of x and y; for, values being assigned to x and y, a value of z is determined.

Similarly, the volume of a cone is a function of its altitude and radius of base; for, the radius and altitude being given, the volume is determined.

2. Kinds of Functions. — An expression containing variables is called an *explicit* function of those variables. Thus $\sqrt{x+y}$ is an explicit function of x and y. Similarly, if

$$y = x + \frac{1}{x}$$

then y is an explicit function of x.

A quantity determined by an equation not solved for that quantity is called an *implicit* function. Thus, if

$$x^2 - 2xy + y^2 = x,$$

y is an implicit function of x. Also x is an implicit function of y.

Explicit and implicit do not denote properties of the function but of the way it is expressed. An implicit function is rendered explicit by solving. For example, the above equation is equivalent to

$$y = x \pm \sqrt{x},$$

in which y appears as an explicit function of x.

A *rational* function is one represented by an algebraic expression containing no fractional powers of variable quantities. For example,

$$\frac{x\sqrt{5} + 3}{x^2 + 2x}$$

is a rational function of x.

A function is called *algebraic* if it can be represented by an algebraic expression or is the solution of an algebraic equation.

Functions that are not algebraic are called *transcendental*. For example, $\sin x$ and $\log x$ are transcendental functions of x.

3. Independent and Dependent Variables. — In most problems there occur a number of variable quantities connected by equations. Arbitrary values can be assigned to some of these quantities and the values of the others are then determined. Those taking arbitrary values are called *independent* variables; those determined are called *dependent* variables. Which variables are taken as independent and which as dependent is usually a matter of convenience. The number of independent variables is however determined by the equations.

For example, in plotting the curve

$$y = x^3 + x$$

values are assigned to x and values of y are calculated. The independent variable is x and the dependent variable y. We might assign values to y and calculate values of x but that would be much more difficult.

4. Notation. — A particular function of x is often represented by the notation $f(x)$, which should be read function of x, or f of x, not f times x. For example

$$f(x) = \sqrt{x^2 + 1}$$

means that $f(x)$ is a symbol for $\sqrt{x^2 + 1}$. Similarly

$$y = f(x)$$

means that y is some definite (though perhaps unknown) function of x.

If it is necessary to consider several functions in the same discussion, they are distinguished by subscripts or accents or by the use of different letters. Thus $f_1(x)$, $f_2(x)$, $f'(x)$, $f''(x)$, $g(x)$ (read f-one of x, f-two of x, f-prime of x, f-second of x, g of x) represent (presumably) different functions of x.

Functions of several variables are represented by writing commas between the variables. For example,

$$v = f(r, h)$$

expresses that v is a function of r and h and

$$v = f(a, b, c)$$

expresses that v is a function of a, b, c.

The f in the symbol of a function should be considered as representing an operation performed on the variable. Thus, if

$$f(x) = \sqrt{x^2 + 1},$$

f represents the operation of squaring the variable, adding 1, and extracting the square root of the result. If x is replaced by any other quantity, the same operation is to be performed on that quantity. For example,

$$f(2) = \sqrt{2^2 + 1} = \sqrt{5},$$
$$f(y + 1) = \sqrt{(y + 1)^2 + 1} = \sqrt{y^2 + 2y + 2}.$$

5. Limit. — If a variable x approaches a constant a in such a way that the difference $x - a$ ultimately becomes and remains numerically less than any preassigned quantity, the constant is said to be the *limit* of the variable. This is expressed by the notation $x \rightarrow a$ or

$$\lim x = a.$$

If a variable x ultimately becomes and remains numerically greater than any preassigned quantity, it is said to *become infinite.* This is expressed by the notation $x \rightarrow \infty$. If the values are ultimately all of one algebraic sign, we sometimes indicate this by writing $x \rightarrow +\infty$, or $x \rightarrow -\infty$, according as they are all positive or all negative.

If $f(x)$ approaches the limit A as x approaches the limit a, this is expressed by the notation

$$\lim_{x \rightarrow a} f(x) = A.$$

Example. Find the value of

$$\lim_{x \rightarrow 1}\left(x + \frac{1}{x}\right).$$

When x approaches the limit 1, it is obvious that $x + \frac{1}{x}$ approaches the limit $1 + \frac{1}{1}$, or 2, in such a way that a stage in the process can be found beyond which the difference of the two remains numerically less than any previously assigned quantity. Hence

$$\lim_{x \to 1}\left(x + \frac{1}{x}\right) = 2.$$

6. Properties of Limits. — In finding the limits of functions frequent use is made of certain simple properties which follow almost immediately from the definition.

1. *The limit of the sum of a finite number of functions is equal to the sum of their limits.*

Suppose, for example, X, Y, Z are three functions approaching the limits A, B, C, respectively. Then $X + Y + Z$ is approaching $A + B + C$. Consequently,

$$\lim (X + Y + Z) = A + B + C = \lim X + \lim Y + \lim Z.$$

2. *The limit of the product of a finite number of functions is equal to the product of their limits.*

If, for example, X, Y, Z approach A, B, C, respectively, then XYZ approaches ABC, that is,

$$\lim XYZ = ABC = \lim X \cdot \lim Y \cdot \lim Z.$$

3. *If the limit of the denominator is not zero, the limit of the quotient of two functions is equal to the quotient of their limits.*

Let X, Y approach the limits A, B and suppose B is not zero. Then $\frac{X}{Y}$ approaches $\frac{A}{B}$, that is,

$$\lim \frac{X}{Y} = \frac{A}{B} = \frac{\lim X}{\lim Y}.$$

If B is zero, the expression

$$\frac{A}{B}$$

has no meaning. In such a case

$$\frac{X}{Y}$$

may or may not approach a limit.

7. The Form $\frac{0}{0}$. — When x is replaced by a particular number a, a function $f(x)$ sometimes assumes the form $\frac{0}{0}$. This symbol does not represent any definite value. Yet the function $f(x)$ may approach a definite limit as x approaches a. This is often made evident by writing the function in a different form.

Example 1. Find the value of

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

When x is replaced by 1 the function takes the form

$$\frac{1 - 1}{1 - 1} = \frac{0}{0}.$$

Since, however,

$$\frac{x^2 - 1}{x - 1} = x + 1,$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim\,(x + 1) = 2.$$

Example 2. Find the value of

$$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}.$$

When $x = 0$ the function becomes

$$\frac{1 - 1}{0} = \frac{0}{0}.$$

Multiplying numerator and denominator by $\sqrt{1+x}+1$, we get

$$\frac{\sqrt{1+x}-1}{x}=\frac{x}{x(\sqrt{1+x}+1)}=\frac{1}{\sqrt{1+x}+1}.$$

As x approaches 0, this last expression approaches $\frac{1}{2}$. Hence

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\tfrac{1}{2}.$$

EXERCISES

1. Given $y^2 - 2\,xy + x^2 - x + 1 = 0$, express y as an explicit function of x.

2. If $f(x) = x^4 - x^2 + 1$, show that $f(-x) = f(x)$.

3. If $f(x) = A \cos x + B \sin x$, show that $f(x + 2\,\pi) = f(x)$.

4. If $f(x, y) = x^2 - 2\,xy$, find $f(y, x)$.

Find the values of the following limits:

5. $\displaystyle\lim_{x\to 0}\frac{x^2+2\,x-3}{x-5}.$

6. $\displaystyle\lim_{x\to 1}\frac{x^2+5\,x-6}{x-1}.$

7. $\displaystyle\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}.$

8. $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\tan\theta}.$

9. When x approaches zero show that $\sin\dfrac{1}{x}$ does not approach a limit.

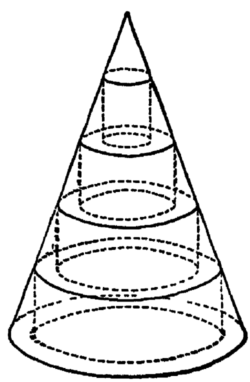

FIG. 7.

10. Inscribe a series of cylinders in a cone, as shown in Fig. 7. When the number of cylinders increases indefinitely, their altitudes approaching zero, does the sum of the volumes of the cylinders approach that of the cone? Does the sum of the lateral areas of the cylinders approach that of the cone?

CHAPTER II

THE DERIVATIVE

8. Increment. — When a variable changes value, the algebraic increase (new value minus old) is called its *increment* and is represented by the symbol Δ written before the variable.

Thus, if x changes from 2 to 4, its increment is

$$\Delta x = 4 - 2 = 2.$$

If x changes from 2 to -1,

$$\Delta x = -1 - 2 = -3.$$

The increment is positive when there is an increase in value, negative when there is a decrease.

Let y be a function of x. When x receives an increment Δx, an increment Δy will be determined. The increments of x and y thus correspond. To illustrate this graphically, let x and y be the rectangular coördinates of a point P. An equation

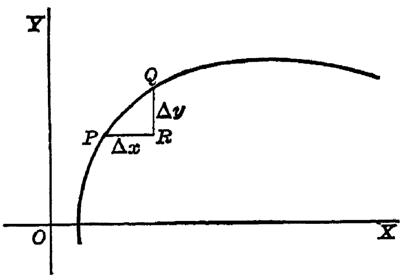

FIG. 8.

$$y = f(x)$$

represents a curve. When x changes the point P changes to some other position Q on the curve. The increments of x and y are

$$\Delta x = PR, \qquad \Delta y = RQ.$$

9. Continuous Function. — A function is called *continuous* if the increment of the function approaches zero as the increment of the variable approaches zero.

In Fig. 8, y is a continuous function of x; for, as Δx approaches zero, Q approaches P and so Δy approaches zero.

In Figs. 9a and 9b are shown two ways that a function can be *discontinuous*. In Fig. 9a the curve has a break at P. As Q approaches P', $\Delta x = PR$ approaches zero but $\Delta y = RQ$ does not. In Fig. 9b the ordinate at $x = a$ is infinite. The increment occurring in a change from $x = a$ to a neighboring value is infinite.

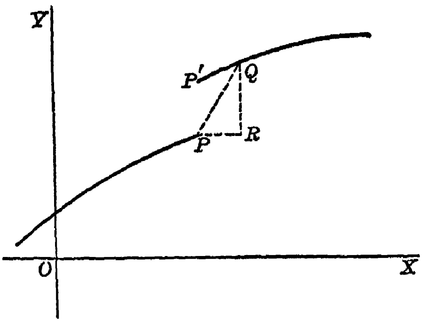

FIG. 9a.

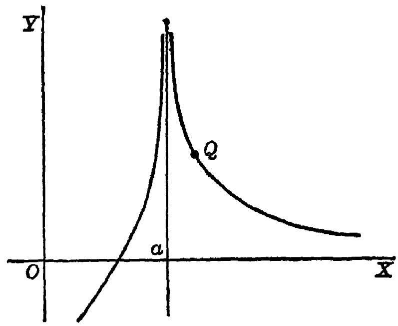

FIG 9b.

10. Slope. — In case of a straight line (Fig. 10a)

$$\frac{\Delta y}{\Delta x} = \frac{RQ}{PR} = \tan \phi \qquad (10a)$$

is called the *slope*. If the line extends upward on the right and downward on the left (as in Fig. 10a) Δx and Δy have the same algebraic sign and the slope is positive. If the line extends downward on the right and upward on the left, Δx and Δy have opposite signs and the slope is negative.

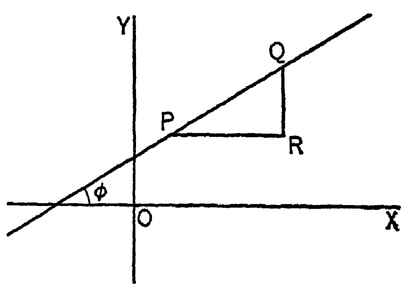

FIG. 10a.

If P and Q are two points on a curve (Fig. 10b),

$$\frac{RQ}{PR} = \frac{\Delta y}{\Delta x}$$

is the slope of the chord PQ. As Q moves along the curve toward P, the line PQ turns about P and usually approaches a limiting position PT. This line PT is called the tangent to the curve at P.

As Q approaches P, Δx approaches zero and the slope of PQ approaches that of PT. Therefore

$$\text{slope of the tangent} = \tan \phi = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{10b}$$

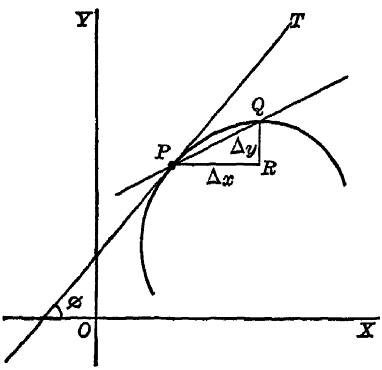

FIG. 10b.

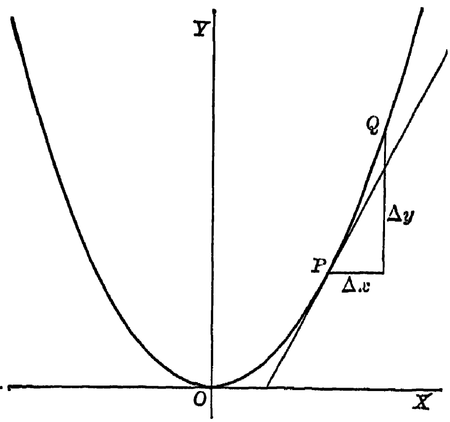

FIG. 10c.

The slope of the tangent at P is called the slope of the curve at P. The angle ϕ is measured from the right end of the x-axis counterclockwise to the tangent line. When this angle is acute the slope is positive, when obtuse the slope is negative.

Example. Find the slope of the parabola $y = x^2$ at the point (1, 1).

Let the coördinates of P be x, y and those of Q, $x + \Delta x$, $y + \Delta y$ (Fig. 10c). Since P and Q are both on the curve,

$$y = x^2$$

and

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2\,x\Delta x + (\Delta x)^2.$$

Subtracting these equations we get

$$\Delta y = 2\,x\Delta x + (\Delta x)^2.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = 2\,x + \Delta x.$$

As Δx approaches zero, this approaches

$$\text{slope at } P = 2\,x.$$

This is the slope at the point $P(x, y)$ with abscissa x. At the point (1, 1) it is then $2.1 = 2$.

11. Derivative. — Let y be a function of x. If $\frac{\Delta y}{\Delta x}$ approaches a limit as Δx approaches zero, that limit is called the *derivative of y with respect to x*. It is represented by the notation $\frac{dy}{dx}$. That is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

The expression $\frac{dy}{dx}$ is at present to be regarded merely as a symbol for the derivative. Later we shall show that it is actually the ratio of two quantities dy and dx called differentials. That is the reason for the notation adopted.

The derivative, being the limit of $\frac{\Delta y}{\Delta x}$, is approximately equal to a small change in y divided by the corresponding small change in x, the approximation becoming better as the increments approach zero.

If small increments of x and y have the same algebraic sign, $\frac{\Delta y}{\Delta x}$ is positive and in the limit $\frac{dy}{dx}$ is positive or zero. If they have opposite signs, $\frac{dy}{dx}$ is negative or zero. Therefore when the derivative is positive x and y increase and decrease to-

gether, when it is negative one variable decreases as the other increases.

Example 1. Given $y = \sqrt[3]{x}$, make a table of corresponding values of Δx and Δy and so estimate the value of $\frac{dy}{dx}$ at $x = 2$.

In the following table values near 2 are assigned to x, the corresponding values of y are calculated, and the increments (new value minus old) are determined using $x = 2$, $y = 1.2599$ as starting point.

x	y	Δx	Δy	$\frac{\Delta y}{\Delta x}$
1.8	1.2164	−0.2	−.0435	0.217
1.9	1.2386	−0.1	−.0213	0.213
2.0	1.2599	0.0	.0000	
2.1	1.2806	0.1	.0207	0.207
2.2	1.3006	0.2	.0407	0.203

As Δx approaches zero either from the positive or the negative side the numbers in the last column seem to approach a common limit approximately 0.210. This is the estimated value of the derivative at $x = 2$.

Example 2. Find the derivative of $y = \frac{1}{x}$.

When x and y receive increments Δx and Δy, the new values are $x + \Delta x$, $y + \Delta y$. Since these satisfy the equation,

$$y + \Delta y = \frac{1}{x + \Delta x}.$$

By subtraction we get

$$\Delta y = \frac{1}{x + \Delta x} - \frac{1}{x} = -\frac{\Delta x}{x(x+\Delta x)}.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = -\frac{1}{x(x + \Delta x)}.$$

When Δx approaches zero, this approaches

$$\frac{dy}{dx} = -\frac{1}{x^2}.$$

EXERCISES

1. Given

$$y = \log_{10} x,$$

from a table of logarithms obtain an approximate value of $\frac{dy}{dx}$ at $x = 10$.

2. Given $y = \sqrt{x}$, make a table of values near $x = 1$ and from this estimate the value of $\frac{dy}{dx}$ at $x = 1$.

In each of the following cases show that the derivative has the value given.

3. $y = x^2 + 3x,\qquad \frac{dy}{dx} = 2x + 3.$

4. $y = x^4,\qquad \frac{dy}{dx} = 4x^3.$

5. $y = \frac{x-1}{x},\qquad \frac{dy}{dx} = \frac{1}{x^2}.$

6. $y = \frac{1}{x^2},\qquad \frac{dy}{dx} = -\frac{2}{x^3}.$

7. Construct the parabola $y = x^2 - 2x$ and find its slope at the point with abscissa x. At what point does the tangent make an angle of 45° with the x-axis?

8. Construct the curve

$$y = x^4 - 2x^2$$

and find the points where the tangent is parallel to the x-axis.

9. If x is an acute angle and $y = \cos x$, does y increase or decrease as x increases? Is $\frac{dy}{dx}$ positive or negative?

10. Show that the slope of the curve

$$y = \frac{1}{x^5}$$

is negative for all values of x.

12. Speed. — If a body moves through equal distances in equal times in a straight or curved path, the distance traveled divided by the time is called its *speed*.

If the body does not travel equal distances in equal times, its speed is variable and cannot be determined in this simple way. In that case the ratio of the distance traveled to the interval of time is called the *average speed* in that interval. Thus

$$\text{average speed} = \frac{\text{distance}}{\text{time}}.$$

During a very short interval of time the speed is nearly constant and becomes more nearly constant as the interval is diminished. Hence the speed at a particular instant is defined as the limit approached by the average speed when the interval containing that instant approaches zero. That is,

$$\text{speed} = \lim \frac{\text{distance}}{\text{time}}.$$

Let s be the distance traveled in time t. Then Δs is the distance traveled in time Δt and

$$\frac{\Delta s}{\Delta t}$$

is the average speed in that interval. Hence the speed at time t is

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}. \tag{12}$$

Thus speed is the derivative of distance traveled with respect to the time.

Example. If the distance a body travels in t seconds is

$$s = 16\,t^2 + 20\,t,$$

find its speed at the end of 2 seconds.

At time $t + \Delta t$ the distance traveled is

$$s + \Delta s = 16\,(t + \Delta t)^2 + 20\,(t + \Delta t).$$

By expansion and subtraction of s we get

$$\Delta s = 32\,t\,\Delta t + 20\,\Delta t + 16\,(\Delta t)^2,$$

whence

$$\frac{\Delta s}{\Delta t} = 32\,t + 20 + 16\,\Delta t.$$

When Δt approaches zero, this approaches

$$v = \frac{ds}{dt} = 32\,t + 20.$$

This is the speed at time t. At time $t = 2$ it is then

$$v = 84 \text{ ft./sec.}$$

EXERCISES

1. A body, starting from rest, falls approximately

$$s = 16\,t^2$$

feet in t seconds Find its average speed during the first 3 seconds and its speed at the end of 3 seconds

2. A body thrown downward with a speed of 100 ft./sec. travels the distance

$$s = 100\,t + 16\,t^2$$

feet in t seconds. Find its average speed between $t = 2$ and $t = 2.01$ and its speed at the time $t = 2$.

3. If the distance a body moves in t seconds is

$$s = t^3 + 2\,t^2,$$

find its speed at the end of t seconds.

4. A particle moving along a straight line travels the distance

$$s = t^3 - 6\,t^2 + 12\,t$$

in t seconds. When does its speed become zero?

5. A particle moves with constant speed in a circle of radius a, making n complete revolutions per second. What is its speed?

CHAPTER III

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

13. Differentiation. — Instead of applying the direct method of the last chapter, differentiation is usually performed by means of certain formulas derived by that method. In this work we use the symbol $\frac{d}{dx}$ for the operation of taking the derivative with respect to x. Thus

$$\frac{d}{dx}(u + v) = \text{derivative of } (u + v) \text{ with respect to } x.$$

14. Formulas. — Let u, v be functions of a single variable x and c, n constants.

I. $\frac{d}{dx}c = 0.$

II. $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$

III. $\frac{d}{dx}(cu) = c\frac{du}{dx}.$

IV. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$

V. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$

VI. $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$

It is assumed that the functions u, v have derivatives. At the values of x used these functions must then be continuous. For, if Δu does not approach zero as Δx approaches zero,

$$\frac{\Delta u}{\Delta x}$$

cannot approach a definite limit.

15. Proof of I. — *The derivative of a constant is zero.*

When the variable x receives an increment Δx, a constant does not vary. Hence $\Delta c = 0$, $\frac{\Delta c}{\Delta x} = 0$, and in the limit $\frac{dc}{dx} = 0$.

16. Proof of II. — *The derivative of the sum of a finite number of functions is equal to the sum of their derivatives.*

Let

$$y = u + v.$$

Since u, v, y are functions of x, when x takes an increment Δx, u changes to $u + \Delta u$, v to $v + \Delta v$, and y to $y + \Delta y$. Consequently

$$y + \Delta y = u + \Delta u + v + \Delta v.$$

Subtraction of the two equations gives

$$\Delta y = \Delta u + \Delta v,$$

whence

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

As Δx approaches zero, $\frac{\Delta y}{\Delta x}, \frac{\Delta u}{\Delta x}, \frac{\Delta v}{\Delta x}$ approach $\frac{dy}{dx}, \frac{du}{dx}, \frac{dv}{dx}$ as limits. If two quantities are always equal, their limits are equal. Hence

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

In a similar way we prove that the derivative of the sum of any finite number of functions is equal to the sum of their derivatives. This is expressed by saying that differentiation is a *distributive* operation, that is, can be applied to (distributed over) the terms of a sum and the results then collected to obtain the derivative of the sum.

17. Proof of III. — *The derivative of a constant times a function is equal to the constant times the derivative of the function.*

Let

$$y = cu.$$

Then

$$y + \Delta y = c(u + \Delta u)$$

and so

$$\Delta y = c\,\Delta u,$$

$$\frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x}.$$

As Δx approaches zero $\dfrac{\Delta y}{\Delta x}$ and $c\,\dfrac{\Delta u}{\Delta x}$ approach $\dfrac{dy}{dx}$ and $c\,\dfrac{du}{dx}$. Therefore

$$\frac{dy}{dx} = c\frac{du}{dx}.$$

If the result of performing two operations one after the other is independent of which is performed first and which second, the operations are called *commutative*. The formula just proved expresses that differentiation and multiplication by a constant are commutative operations.

A fraction with a constant denominator should be differentiated by this formula. Thus

$$\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{d}{dx}\left(\frac{1}{c}\cdot u\right) = \frac{1}{c}\frac{du}{dx}.$$

18. Proof of IV. — *The derivative of the product of two functions is equal to the first times the derivative of the second plus the second times the derivative of the first.*

Let

$$y = uv.$$

Then

$$\begin{aligned} y + \Delta y &= (u + \Delta u)\,(v + \Delta v) \\ &= uv + v\Delta u + (u + \Delta u)\,\Delta v. \end{aligned}$$

Subtraction gives

$$\Delta y = v\,\Delta u + (u + \Delta u)\,\Delta v,$$

whence

$$\frac{\Delta y}{\Delta x} = v\,\frac{\Delta u}{\Delta x} + (u + \Delta u)\,\frac{\Delta v}{\Delta x}.$$

Since u is a continuous function, Δu approaches zero as Δx approaches zero. Therefore in the limit

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

In a similar way we can prove

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}.$$

The derivative of a product is thus obtained by differentiating the factors one at a time, multiplying by the product of the other factors, and adding the results.

It is to be noted that differentiation and multiplication by a variable are not commutative operations. That is,

$$\frac{d}{dx}(uv)$$

and

$$u\frac{dv}{dx}$$

are not in general equal.

19. Proof of V. — *The derivative of a fraction is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator,* all divided by *the square of the denominator.*

Let

$$y = \frac{u}{v}.$$

Then

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$$

and

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\,\Delta u - u\,\Delta v}{v(v + \Delta v)}.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}.$$

Since v is a continuous function of x, Δv approaches zero as Δx approaches zero. Therefore in the limit

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

This proof is valid however small v may be provided it is not zero. As v approaches zero, if both sides of the equation approach definite limits, those limits must be equal.

20. Proof of VI. — *The derivative of a variable raised to a constant power is equal to the product of the exponent, the variable raised to a power one less, and the derivative of the variable.*

We consider three cases depending on whether the exponent is a positive whole number, a positive fraction, or a negative rational number.

(1) Let n be a positive integer and $y = u^n$. Then by the binomial theorem

$$\begin{aligned} y + \Delta y &= (u + \Delta u)^n \\ &= u^n + nu^{n-1}\,\Delta u + \frac{n(n-1)}{2}u^{n-2}\,(\Delta u)^2 + \cdots. \end{aligned}$$

Consequently,

$$\Delta y = nu^{n-1}\,\Delta u + \frac{n(n-1)}{2}u^{n-2}\,(\Delta u)^2 + \cdots$$

and

$$\frac{\Delta y}{\Delta x} = \left[nu^{n-1} + \frac{n(n-1)}{2} u^{n-2}\,\Delta u + \cdots \right] \frac{\Delta u}{\Delta x}.$$

As Δx approaches zero, Δu approaches zero. Hence in the limit

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx},$$

which proves the formula in case n is a positive integer.

(2) Let n be a positive fraction $\frac{p}{q}$ and

$$y = u^n = u^{\frac{p}{q}}.$$

Then

$$y^q = u^p.$$

Since p and q are both positive integers, we can apply the formula just proved in (1) and so obtain

$$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}.$$

Solving for $\frac{dy}{dx}$ we get

$$\frac{dy}{dx} = \frac{p}{q}\frac{u^{p-1}}{y^{q-1}}. \quad \frac{du}{dx}.$$

Since $y = u^{\frac{p}{q}}$, we have

$$y^{q-1} = u^{\frac{p}{q}(q-1)} = u^{p-\frac{p}{q}}.$$

Hence

$$\frac{dy}{dx} = \frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}.$$

(3) Let n be a negative rational number $-m$. Then

$$y = u^n = u^{-m} = \frac{1}{u^m}.$$

Since m is positive, we can find the derivative of u^m by (2). Therefore, by V

$$\frac{dy}{dx} = \frac{u^m \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx} u^m}{(u^m)^2} = -\frac{mu^{m-1}\frac{du}{dx}}{u^{2m}}$$
$$= -mu^{-m-1}\frac{du}{dx} = nu^{n-1}\frac{du}{dx}.$$

Therefore whether n is an integer or fraction, positive or negative,

$$\frac{d}{dx} u^n = nu^{n-1}\frac{du}{dx}.$$

Example 1. $y = 4\,x^3$.
Using formulas III and VI,

$$\frac{dy}{dx} = 4\frac{d}{dx}(x^3) = 4\,(3\,x^2) = 12\,x^2.$$

Example 2. $y = x^4 - 3\,x^2 + 6\,x + 7$.
Differentiating term by term,

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) - 3\frac{d}{dx}(x^2) + 6\frac{d}{dx}(x) + \frac{d}{dx}(7)$$
$$= 4\,x^3 - 6\,x + 6.$$

Example 3. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$.

This can be written

$$y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}.$$

Consequently, by II and VI,

$$\frac{dy}{dx} = \frac{d}{dx}(x^{\frac{1}{2}}) + \frac{d}{dx}(x^{-\frac{1}{2}})$$
$$= \tfrac{1}{2}\,x^{-\frac{1}{2}} - \tfrac{1}{2}\,x^{-\frac{3}{2}}$$
$$= \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}.$$

Example 4. $y = (x + a)(x^2 - b)$.

It is assumed that a and b are constant. Using IV with $u = x + a$, $v = x^2 - b$,

$$\begin{aligned}\frac{dy}{dx} &= (x + a)\frac{d}{dx}(x^2 - b) + (x^2 - b)\frac{d}{dx}(x + a)\\ &= (x + a)(2x - 0) + (x^2 - b)(1 + 0)\\ &= 3x^2 + 2ax - b.\end{aligned}$$

Example 5. $y = \dfrac{x^2 - 1}{x^2 + 1}$.

Using V, with $u = x^2 - 1$, $v = x^2 + 1$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2 + 1)\dfrac{d}{dx}(x^2 - 1) - (x^2 - 1)\dfrac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2}\\ &= \frac{(x^2 + 1)\,2x - (x^2 - 1)\,2x}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}.\end{aligned}$$

Example 6. $y = \sqrt{x^2 - 1}$.

Using VI, with $u = x^2 - 1$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2 - 1)^{\frac{1}{2}} = \tfrac{1}{2}(x^2 - 1)^{-\frac{1}{2}}\frac{d}{dx}(x^2 - 1)\\ &= \tfrac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2 - 1}}\end{aligned}$$

EXERCISES

Find $\dfrac{dy}{dx}$ in each of the following exercises:

1. $y = 3x^2 - 2x + 3$.
2. $y = x^4 - 4x^3 + 6x^2 - 5$.
3. $y = \frac{1}{3}(x^3 + 3x^2 - 5)$.
4. $y = 2\sqrt{x^3} + 6\sqrt{x}$.
5. $y = (x - 2a)(a^2 - x^2)$.
6. $y = \sqrt{x}(2x - 1)$.
7. $y = x(3x - 1)^2$.
8. $y = (x + 1)(3x - 2)^2(2x + 3)^3$.
9. $y = \dfrac{1}{1 - x^2}$.
10. $y = \dfrac{2x - 1}{2x + 3}$.
11. $y = \dfrac{3x - 1}{(x - 1)^3}$.
12. $y = \sqrt{1 + 2x - x^2}$.
13. $y = \dfrac{\sqrt{a^2 - x^2}}{x}$.
14. $y = x\sqrt{a^2 - x^2}$.
15. $y = \dfrac{a}{x + \sqrt{a^2 + x^2}}$.
16. $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$.

17. $y = \dfrac{2\,ax + b}{\sqrt{ax^2 + bx + c}}.$

18. $y = x(4\,x^2 - 5)\sqrt{2\,x^2 - 1}.$

19. $y = \dfrac{2\,x^2 - 1}{3\,x^3}\sqrt{x^2 + 1}.$

20. Find the slope of the curve

$$y = \frac{x}{\sqrt{x^2 + 9}}$$

at the point $x = 4$.

21. Find the slope of the curve

$$y = x(x^5 + 31)^{\frac{4}{5}}$$

at the point $x = 1$.

22. Find the points on the curve

$$y = (x - 1)^3\,(x + 2)^4$$

where the tangent is parallel to the x-axis.

23. Find the angle between the x-axis and the tangent to the curve

$$y = x\sqrt{1 - x^2}$$

at the origin.

21. Higher Derivatives. — The first derivative $\dfrac{dy}{dx}$ is a function of x. Its derivative with respect to x, written $\dfrac{d^2y}{dx^2}$ is called the *second derivative* of y with respect to x, or derivative of the *second order*. That is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Similarly, the third derivative, or derivative of the third order, is

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right), \text{ etc.}$$

For example, if

$$y = x^3,$$

we find

$$\frac{dy}{dx} = 3\,x^2,$$

$$\frac{d^2y}{dx^2} = 6\,x,$$

$$\frac{d^3y}{dx^3} = 6,$$

$$\frac{d^4y}{dx^4} = 0.$$

All higher derivatives are zero.

The second derivative

$$\frac{d^2y}{dx^2}$$

and the square of the first derivative

$$\left(\frac{dy}{dx}\right)^2$$

should be carefully distinguished. Inspection of the values in the above example will show that these are not in general equal. The notation

$$\frac{d^ny}{dx^n}$$

indicates that in the process of forming the nth derivative y is introduced in the numerator only once, whereas x is introduced in the denominator (through division by Δx) as many times as differentiation is performed.

22. Differentiation of Implicit Functions. — If x and y satisfy an algebraic equation, this can sometimes be solved for y and the derivative then obtained. It may however be simpler to differentiate the equation term by term and solve the resulting equation for the derivative.

Suppose, for example,

$$x^2 + xy + y^2 = 1. \tag{22a}$$

Differentiating with respect to x,

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0,$$

whence

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y}. \tag{22b}$$

To get the second derivative, we differentiate again with respect to x and so obtain

$$\frac{d^2y}{dx^2} = \frac{(2x + y)\left(1 + 2\frac{dy}{dx}\right) - (x + 2y)\left(2 + \frac{dy}{dx}\right)}{(x + 2y)^2}$$

$$= \frac{3x\frac{dy}{dx} - 3y}{(x + 2y)^2}.$$

Replacing $\frac{dy}{dx}$ by its value from (22b) and reducing,

$$\frac{d^2y}{dx^2} = -\frac{6(x^2 + xy + y^2)}{(x + 2y)^3}.$$

Since x and y satisfy (22a), this can be further reduced to the form

$$\frac{d^2y}{dx^2} = -\frac{6}{(x + 2y)^3}.$$

By differentiating this again with respect to x we could find the third derivative, etc.

EXERCISES

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in each of the following cases:

1. $y = x - \frac{1}{x}$.
2. $y = \frac{x - 1}{x + 1}$.
3. $y = x^2(x + 2)^3$.
4. $y^2 = ax$.
5. $x^2 + y^2 = a^2$.
6. $x^2 - y^2 = 1$.
7. $xy = x + y$.
8. $x^2 - 2xy = 1$.

9. $x^2 - xy + y^2 = 0.$

10. Given

$$x^3 + y^3 = 3\,xy,$$

find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ and show that

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}.$$

11. If a, b, c, d are constants and

$$y = ax^3 + bx^2 + cx + d,$$

show that

$$\frac{d^4y}{dx^4} = 0.$$

12. Find the derivative of

$$t^2\frac{d^2x}{dt^2} - 2\,t\frac{dx}{dt} + 2\,x$$

with respect to t.

13. Why can we not differentiate the equation

$$x^2 - 4\,x + 3 = 0$$

term by term and so obtain $2\,x - 4 = 0$?

CHAPTER IV

RATES

23. Rate of Change. — If the change in a quantity z is proportional to the time in which it occurs, z is said to change at a constant rate. If Δz is the change occurring in an interval of time Δt, the rate of change of z is

$$\frac{\Delta z}{\Delta t}.$$

If the rate of change is not constant, it is usually nearly constant when the interval Δt is very short. Then $\dfrac{\Delta z}{\Delta t}$ is approximately the rate of change of z, the approximation becoming better as the interval decreases. The exact rate of change at time t is consequently defined as

$$\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}; \tag{23}$$

that is, *the rate of change of any quantity is its derivative with respect to the time.*

If the quantity z is increasing, its rate of change is positive; if decreasing, the rate is negative.

24. Velocity Along a Straight Line. — Let a particle P move along a straight line (Fig. 24). Let the distance

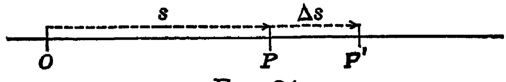

FIG. 24.

$s = OP$ be considered positive on one side of O and negative on the other. If the particle describes the segment $PP' = \Delta s$ in time Δt the quantity

$$\frac{\Delta s}{\Delta t}$$

is called the average velocity in that interval. The velocity at P is defined as the limit approached by this average velocity as the interval is diminished indefinitely. That is

$$\text{velocity} = v = \frac{ds}{dt}. \tag{24}$$

When s is (algebraically) increasing the velocity is positive; when decreasing, the velocity is negative. The algebraic sign of the velocity thus merely indicates whether or not the motion is in the direction which has been taken as positive for s.

The numerical value of the velocity (without the algebraic sign) is called speed. Thus speed is the rate of change of distance traveled without regard to direction of motion.

25. Acceleration Along a Straight Line. — The acceleration of a particle moving along a straight line is defined as the rate of change of its velocity. That is

$$\text{acceleration} = a = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \tag{25}$$

The acceleration is positive when the velocity is (algebraically) increasing; negative, when it is decreasing.

Example. At the end of t seconds the vertical height of a ball thrown upward is

$$h = 100\,t - 16\,t^2.$$

Find its velocity and acceleration. Also find when it is rising, when falling, and when it reaches the highest point.

The velocity and acceleration are

$$v = \frac{dh}{dt} = (100 - 32\,t) \text{ ft./sec.}$$

$$a = \frac{dv}{dt} = -32 \text{ ft./sec.}^2$$

The ball will be rising while v is positive, that is, until

$$\frac{100}{32} = 3\tfrac{1}{8}.$$

It will be falling after $t = 3\frac{1}{8}$. It will be at the highest point when $t = 3\frac{1}{8}$.

26. Angular Velocity and Acceleration. — Consider a body rotating about a fixed axis. Let θ be the angle turned through at time t. The angular velocity of the body is defined as the rate of change of θ. That is

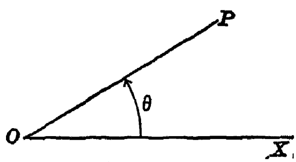

FIG. 26.

$$\text{angular velocity} = \omega = \frac{d\theta}{dt}. \quad (26a)$$

The angular acceleration of the body is defined as the rate of change of its angular velocity. That is

$$\text{angular acceleration} = \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \quad (26b)$$

In work that involves differentiation, angles are usually measured in radians.

Example. A wheel, starting from rest under the action of a constant torque about its axis, will turn in t seconds through an angle

$$\theta = kt^2,$$

k being constant. Find its angular velocity and acceleration.

By definition

$$\omega = \frac{d\theta}{dt} = 2\,kt \text{ rad./sec.}$$

$$\alpha = \frac{d\omega}{dt} = 2\,k \text{ rad./sec.}^2$$

27. Related Rates. — In many cases the rates of change of certain variables are known and the rates of others are to be calculated. This is done by writing the equations connecting the variables and differentiating these with respect to the time. The resulting equations will contain both the known and the unknown rates. When values are substituted for the known rates, if the problem is determinate,

there will be equations enough to solve for the unknown ones.

Example 1. The radius of a cylinder is increasing 2 ft./sec. and its altitude decreasing 3 ft./sec. Find the rate of change of its volume.

Let r be the radius and h the altitude. Then the volume is

$$v = \pi r^2 h.$$

The rate of change of volume is

$$\frac{dv}{dt} = \pi r^2 \frac{dh}{dt} + 2\,\pi r h \frac{dr}{dt}.$$

By hypothesis,

$$\frac{dr}{dt} = 2, \qquad \frac{dh}{dt} = -3.$$

Hence

$$\frac{dv}{dt} = 4\,\pi r h - 3\,\pi r^2.$$

This is the rate of increase when the radius is r and the altitude h. If $r = 10$ ft., $h = 6$ ft.,

$$\frac{dv}{dt} = -60\,\pi \text{ ft./sec.}$$

Example 2. A ship B sailing south 16 miles per hour is northwest of a ship A sailing east 10 miles per hour. At what rate are the ships approaching?

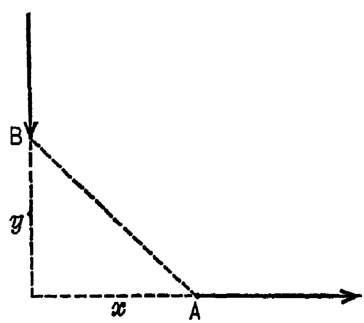

FIG. 27.

Let x and y be the distances of the ships A and B from the point where their paths cross. The distance between the ships is then

$$s = \sqrt{x^2 + y^2}.$$

The rate of change of s is the rate at which the ships are approaching or separating. By differentiation we find

$$\frac{ds}{dt} = \frac{x\dfrac{dx}{dt} + y\dfrac{dy}{dt}}{\sqrt{x^2 + y^2}}.$$

By hypothesis

$$\frac{dx}{dt} = 10, \qquad \frac{dy}{dt} = -16,$$

$$\frac{x}{\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}} = \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

Therefore

$$\frac{ds}{dt} = \frac{10 - 16}{\sqrt{2}} = -3\sqrt{2} \text{ mi./hr.}$$

The negative sign shows that s is decreasing, that is, the ships are approaching.

EXERCISES

1. A ball thrown upward reaches the height

$$h = 20 + 80\,t - 16\,t^2$$

at the end of t seconds. Find its velocity and acceleration when $t = 2$. How long does it continue to rise? What is the highest point reached?

2. A particle moves along a straight line the distance

$$s = \tfrac{1}{3}\,t^3 - 16\,t$$

in t seconds. Find its acceleration at the point where its velocity becomes zero.

3. A particle moves along a straight line according to the law

$$s = t^3 - 6\,t^2 - 6\,t + 20.$$

During what interval does its velocity decrease? Does its speed decrease during that interval?

4. If the distance a particle moves in t seconds is

$$s = t^{\frac{3}{2}},$$

show that it must start with infinite acceleration.

5. A body, starting from rest, has the velocity

$$v = 8\sqrt{h}$$

when it has fallen h ft. By differentiating with respect to the time show that its acceleration is 32 ft./sec.

6. A wheel is turning 100 revolutions per minute. What is its angular velocity? If the wheel is 4 ft. in diameter, with what speed does it drive a belt?

7. Two pulleys, diameters 2 and 4 ft., are connected by a belt. What is the ratio of their angular velocities and which is greater?

8. A wheel of radius r rolls along a line. If s is the distance moved by the center and θ the angle turned through at time t, show that

$$s = r\theta$$

If v is the velocity and a the acceleration of the center, ω the angular velocity and α the angular acceleration about its axis, show that

$$v = r\omega, \qquad a = r\alpha.$$

9. A wheel of radius r rolls down an inclined plane, its center moving the distance $s = 5\,t^2$ in t seconds. Find the angular acceleration of the wheel about its axis.

10. A stone dropped into a pond sends out a series of concentric ripples. If the radius r of the outer ripple increases steadily at the rate of v ft./sec., find the rate at which the area of disturbed water is increasing

11. At a certain instant the altitude of a cone is 3 ft. and the radius of its base is 2 ft If the altitude is increasing 1 ft./sec. and the radius of base decreasing 0.5 ft./sec., find the rate of change of volume.

12. A kite is 300 ft. high and there are 500 ft. of cord out. Assuming the cord to stretch in a straight line, if the kite moves horizontally at the rate of 5 miles per hour directly away from the person flying it, how fast is the cord being paid out?

13. The top of a ladder 20 ft. long slides down a vertical wall. Find the ratio of the speeds of the top and bottom when the ladder makes an angle of 60° with the ground.

14. The cross section of a trough 6 ft long is an equilateral triangle. If water flows in at the rate of 2 cu. ft./sec., find the rate at which the depth is increasing when the water is 18 inches deep.

15 Water flows from a conical funnel at a rate proportional to the square root of the depth h. Show that the rate of change of depth is

$$\frac{dh}{dt} = \frac{k}{h^{\frac{3}{2}}},$$

where k is constant.

16. A man 6 ft. tall walks at the rate of 5 ft./sec. away from a lamp 15 ft. above the ground. When he is 20 ft. from the lamp post find the rate at which the end of his shadow is moving and the rate at which his shadow is growing.

17. A particle moves along the parabola

$$y^2 = 2\,px$$

in such a way that its projection on the y-axis has a constant velocity. Show that its projection on the x-axis moves with constant acceleration.

18. At a certain time a ship A is 20 miles east of a ship B. If A sails north 8 miles per hour and B south 12 miles per hour, how fast are they separating 2 hours later?

19. Two straight railway tracks intersect at an angle of 60° On one a train is 8 miles from the junction and moving toward it at the rate of 40 miles per hour. On the other a train is 12 miles from the junction and moving from it at the rate of 20 miles per hour. Find the rate at which the trains are approaching or separating.

20. The rays of the sun make an angle of 30° with the ground. A ball drops from a height of 64 ft. Assuming that it falls

$$s = 16\,t^2 \text{ ft.}$$

in t seconds, find the speed of its shadow on the ground just before it strikes.

21. The side of an equilateral triangle is increasing at the rate of 10 ft. per minute and its area at the rate of 100 sq. ft. per minute. Find the side of the triangle.

CHAPTER V

MAXIMA AND MINIMA

28. General Theory. — A function $f(x)$ is said to have a *maximum* at $x = a$ if when $x = a$ the function is greater than for any other value of x in the immediate neighborhood of a. It has a *minimum* if when $x = a$ the function is less than for any other value of x sufficiently near a.

If we represent the function by y and plot the curve

$$y = f(x)$$

a maximum occurs at the top, a minimum at the bottom of a wave. Thus, in Fig. 28a, the function has a maximum at A and a minimum at B. It should be noted that a maximum is not necessarily the greatest value and a minimum not necessarily the least value of the function.

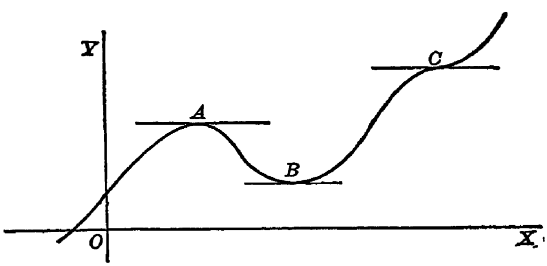

FIG. 28a.

If the derivative is continuous, as in Fig. 28a, the tangent is horizontal at the highest and lowest points of a wave and the slope is zero. Hence in determining maxima and minima of a function $f(x)$, we first look for values of x such that

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = 0.$$

Represent the derivative of $f(x)$ by $f'(x)$. The equation just written is then

$$f'(x) = 0.$$

If a is a root of this equation, $f(a)$ may be a maximum value of $f(x)$, a minimum, or neither.

If the slope is positive on the left of the point and negative on the right, as at A, the curve falls on both sides and the ordinate has a maximum value at that point. That is, *$f(x)$ has a maximum value at $x = a$ if $f'(x)$ is positive for values of x a little less and negative for values a little greater than a.*

If the slope is negative on the left and positive on the right, as at B, the curve rises on both sides and the ordinate has a minimum value at the point. That is, *$f(x)$ has a minimum value at $x = a$ if $f'(x)$ is negative for values of x a little less and positive for values a little greater than a.*

If the slope has the same sign on both sides, as at C, the curve rises on one side and falls on the other and the ordinate is neither a maximum nor a minimum. That is, *$f(x)$ has neither a maximum nor a minimum at $x = a$ if $f'(x)$ has the same sign on both sides of $x = a$.*

Example 1. The sum of two numbers is 5. Find the maximum value of their product.

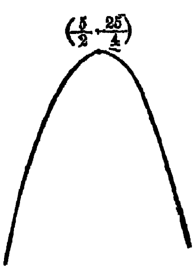

FIG. 28b.

Let one of the numbers be x. The other is then $5 - x$. The value of x is to be found such that the product

$$y = x(5 - x) = 5\,x - x^2$$

is a maximum. The derivative is

$$\frac{dy}{dx} = 5 - 2\,x.$$

This is zero when $x = \frac{5}{2}$. If x is less than $\frac{5}{2}$, the derivative is positive. If x is greater than $\frac{5}{2}$, the derivative is negative.

Near $x = \frac{5}{2}$ the graph then has the shape shown in Fig. 28b. At $x = \frac{5}{2}$ the function then has its maximum value

$$y = x(5 - x) = \frac{5}{2}\left(5 - \frac{5}{2}\right) = \frac{25}{4}.$$

Example 2. Find the shape of a quart can, open at the top, which requires for its construction the least amount of tin.

Let the radius of the base be r and the depth h. The area of the base is πr^2 and that of the side wall $2\,\pi r h$. Hence the area of tin used is

$$A = \pi r^2 + 2\,\pi r h.$$

Let v be the number of cubic inches in a quart. Then

$$v = \pi r^2 h.$$

Consequently

$$h = \frac{v}{\pi r^2}$$

and

$$A = \pi r^2 + \frac{2\,v}{r}.$$

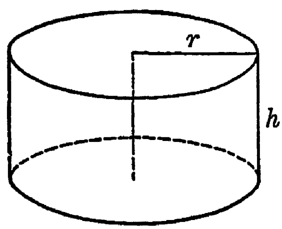

FIG. 28c.

Since π and v are constants

$$\frac{dA}{dr} = 2\,\pi r - \frac{2\,v}{r^2} = 2\left(\frac{\pi r^3 - v}{r^2}\right).$$

This is zero when $\pi r^3 = v$, that is, when

$$r = \sqrt[3]{\frac{v}{\pi}}.$$

If a smaller value than this is substituted for r, πr^3 will be less than v and $\frac{dA}{dr}$ will be negative. If a larger value is

substituted for r, πr^3 will be greater than v and $\frac{dA}{dr}$ positive. Hence the solution obtained is a minimum. Another method of showing this is to observe that since the amount of tin cannot be zero there must be a least amount. For this least value the derivative must be zero (see exceptions in Art. 30). The derivative is zero for only one value of r. This value of r must then give the minimum.

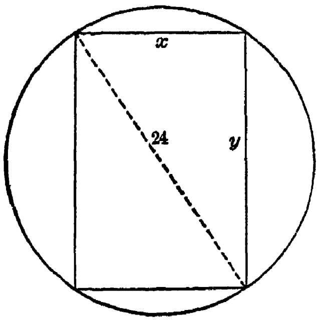

FIG. 28d

Combining the equation

$$\pi r^3 = v,$$

which determines the minimum, with the equation

$$v = \pi r^2 h,$$

we see that $r = h$. The can requiring the least amount of tin thus has a radius equal to its depth.

Example 3. The strength of a rectangular beam is proportional to the product of its width by the square of its depth. Find the strongest beam of rectangular section that can be cut from a log 24 inches in diameter.

In Fig. 28d is shown a section of the log and beam. Let x be the breadth and y the depth of the beam. Then

$$x^2 + y^2 = 24^2.$$

The strength of the beam is

$$s = kxy^2 = kx(24^2 - x^2).$$

Hence

$$\frac{ds}{dx} = k(24^2 - 3\,x^2).$$

This is zero if

$$x = \pm 8\sqrt{3}.$$

Since x is the breadth of the beam it cannot be negative. Hence

$$x = 8\sqrt{3}$$

is the only solution. By testing the signs of the derivative we can show that this determines a maximum value of s. Or, we can say that, since the beam cannot be infinitely strong, there must be a strongest beam. Since no other value can give either a maximum or a minimum $x = 8\sqrt{3}$ must be the width of that strongest beam. From the equation

$$x^2 + y^2 = 24^2$$

the corresponding depth is found to be

$$y = 8\sqrt{6}.$$

29. Second Derivative Test. — Let $f''(x)$ be the second derivative of $f(x)$. Then

$$f''(x) = \frac{d}{dx}f'(x).$$

Consequently, when $f''(x)$ is positive $f'(x)$ increases as x increases and when $f''(x)$ is negative $f'(x)$ decreases as x increases.

Suppose

$$f'(a) = 0.$$

If $f''(a)$ is positive, $f'(x)$ increases through zero as x increases through a. Hence $f'(x)$ must be negative for values of x a little less than a and positive for values a little greater than a. The function then has a minimum value at $x = a$.

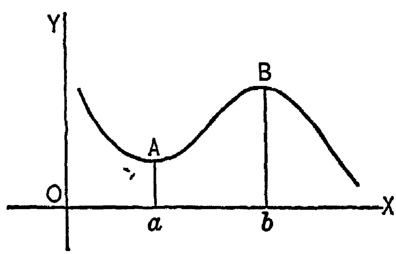

FIG. 29a.

If $f'(a) = 0$ and $f''(a)$ is negative, $f'(x)$ decreases through zero as x increases through a. Hence $f'(x)$ must be positive for values of x a little less than a and negative for values a

little greater than a. The function then has a maximum at $x = a$.

Therefore, *if* $f'(a) = 0$ *and* $f''(a) > 0$, *the function* $f(x)$ *has a minimum value at* $x = a$. *If* $f'(a) = 0$ *and* $f''(a) < 0$, *the function* $f(x)$ *has a maximum at* $x = a$.

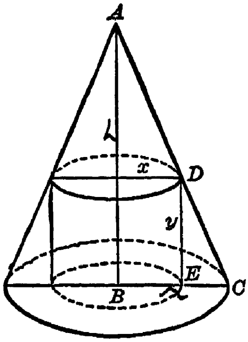

FIG. 29b

This test fails when $f''(a)$ is zero or discontinuous at $x = a$. In other cases it may be used instead of the test in Art. 28 if desired.

Example. Find the dimensions of the largest right circular cylinder that can be inscribed in a given right circular cone.

Let r be the radius and h the altitude of the cone, x the radius and y the altitude of the inscribed cylinder. From the similar triangles DEC and ABC (Fig. 29b),

$$\frac{DE}{EC} = \frac{AB}{BC},$$

that is,

$$\frac{y}{r - x} = \frac{h}{r}, \qquad y = \frac{h}{r}(r - x).$$

The volume of the cylinder is

$$v = \pi x^2 y = \pi \frac{h}{r}(rx^2 - x^3).$$

Hence

$$\frac{dv}{dx} = \frac{\pi h}{r}(2\,rx - 3\,x^2).$$

This is zero when $x = 0$ or $x = \frac{2}{3}\,r$. The value $x = 0$ obviously does not give the maximum. Hence

$$x = \tfrac{2}{3}\,r$$

is the only value to try. The second derivative of v is

$$\frac{d^2v}{dx^2} = \frac{\pi h}{r}(2\,r - 6\,x).$$

Substituting $x = \frac{2}{3}\,r$, we find

$$\frac{d^2v}{dx^2} = -\,2\,\pi h$$

Since the first derivative is zero and the second derivative negative, the value of x found determines a maximum volume.

30. Method of Finding Maxima and Minima. — The method used in solving these problems involves the following steps:

(1) Decide what is to be a maximum or minimum. Let it be y.

(2) Express y in terms of a *single* variable. Let it be x.

It may be convenient to express y temporarily in terms of several variable quantities. If the problem can be solved by our present methods, there will be relations enough to eliminate all but one of these.

(3) Calculate $\frac{dy}{dx}$ and find for what values of x it is zero.

(4) It is usually easy to decide from the problem itself whether the corresponding values of y are maxima or minima. If not determine the signs of $\frac{dy}{dx}$ when x is a little less and a little greater than the values in question and apply the criteria of Art. 28, or find the second derivative and apply the criteria of Art. 29.

EXERCISES

Find the maximum and minimum values of the following functions:

1. $x^2 - 4\,x + 6.$
2. $2\,x^3 + 3\,x^2 - 12\,x.$
3. $x^4 - 2\,x^2 + 6.$
4. $\dfrac{x^2}{\sqrt{a^2 - x^2}}.$

Show that the following functions have no maxima or minima:

5. $x^5 - 1$. **7.** $x^3 - 6x^2 + 12x + 4$.

6. $x^3 + 3x$. **8.** $x\sqrt{a^2 + x^2}$.

9. Show that $x + \frac{1}{x}$ has a maximum and a minimum but that the maximum is less than the minimum

10. Show that the largest rectangle with a given perimeter is a square.

11. Show that the largest rectangle that can be inscribed in a given circle is a square.

12. Find the altitude of the largest right circular cylinder that can be inscribed in a sphere of radius a.

13. Find the dimensions of the largest right circular cone inscribed in a sphere of radius a.

14. A rectangular box with square base and open at the top is to be made from a given amount of material. If no allowance is made for thickness of material or waste in construction, what are the dimensions of the largest box that can be made?

15. A cylindrical tin can closed at both ends is to have a given capacity. Show that the amount of tin required will be a minimum when the height equals the diameter

16. The top, bottom, and lateral surface of a closed tin can are to be cut from rectangles of tin, the scraps being a total loss. Find the most economical proportions for a can of given capacity

17. A box is to be made out of a piece of cardboard, 6 inches square, by cutting equal squares from the corners and turning up the sides. Find the dimensions of the largest box that can be made in this way.

18. Find the volume of the largest right cone that can be generated by rotating a right triangle of hypotenuse 2 ft. about one of its sides.

19. Among all circular sectors with a given perimeter, find the one which has the greatest area.

20. If the sum of the length and girth of a package must not exceed 72 inches, find the dimensions of the largest package of rectangular cross section.

21. The same distance was measured 4 times, the results being a_1, a_2, a_3, a_4. By the theory of least squares the most probable value for the correct distance is that which makes the sum of the squares of the four errors a minimum. What is that value?

22. The sides of a trough of triangular section are planks 12 inches wide. Find the width at the top if the trough has maximum capacity.

23. A gutter of rectangular section is to be made by bending into shape a rectangular strip of copper. Show that the capacity of the gutter will be greatest if its width is twice its depth.

24. A gutter of trapezoidal section is made by joining three flat strips each 4 inches wide, the middle one being horizontal and the other two inclined at the same angle. How wide should the gutter be at the top to have maximum capacity?

25. A circular filter paper of radius 3 inches is to be folded into a conical filter. Find the radius of the base of the filter if it has maximum capacity.

26. A window has the form of a rectangle surmounted by a semicircle. If the perimeter is 40 ft., find the dimensions so that the greatest amount of light may be admitted.

27. What are the most economical dimensions of an open cylindrical water tank if the cost of the sides per square foot is two-thirds that of the bottom?

28. If the top and bottom margins of a printed page are each of width a, the side margins of width b, and the text covers an area c, what should be the dimensions of the page to use the least paper?

29. To reduce the friction of a liquid against the walls of a channel, the channel should be so designed that the area of wetted surface is as small as possible. Show that the best form for an open rectangular channel of given area is that in which the width equals twice the depth.

30. Find the dimensions of the best trapezoidal channel of given area if the banks make an angle θ with the vertical.

31. Find the least area of canvas that can be used to make a conical tent of 1000 cu. ft. capacity.

32. Find the maximum capacity of a conical tent made from 1000 sq. ft. of canvas

33. Find the dimensions of the smallest right cone that can contain a sphere of radius a.

34. Assuming that the intensity of light is inversely proportional to the square of the distance from the source, find the point on the line joining two sources, one of which is twice as intense as the other, at which the illumination is a minimum.

35. A ship B is 75 miles due east of a ship A. If B sails west 12 miles per hour and A south 9 miles per hour, find when the ships will be closest together.

36. A man on one side of a river 1 mile wide wishes to reach a point on the opposite side 5 miles further up the river. If he can walk 4 miles per hour and row 2 miles, find the route he should take to make the trip in least time.

37. A fence 6 ft. high runs parallel to and 5 ft. from a wall. Find the shortest ladder that will reach from the ground over the fence to the wall.

38. A log has the form of a frustum of a cone 30 ft. long, the diameters of its ends being 2 ft. and 1 ft. A beam of square section is to be cut from the log. Find its length if the volume of the beam is a maximum.

39. A and C are points on the same side of a plane mirror. A ray of light passes from A to C by way of a point B on the mirror. Show that the length of the path will be least when the lines AB, BC make equal angles with the perpendicular to the mirror.

40. Let the velocity of light in air be v_1 and in water v_2. The path of a ray of light from a point A in the air to a point C below the surface of the water is bent at B where it enters the water. If θ_1 and θ_2 are the angles which AB and BC make with the perpendicular to the surface, show that the time required for light to pass from A to C will be least if B is so placed that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}.$$

41. The cost per hour of propelling a steamer is proportional to the cube of her speed through the water. Find the speed at which a boat should be run against a current of 5 miles per hour to make a given trip at least cost.

42. If the cost per hour for fuel required to run a steamer is proportional to the cube of her speed and is \$20 per hour for a speed of 10 knots, and if the other expenses amount to \$100 per hour, find the most economical speed in still water.

31. Other Types of Maxima and Minima. — The method given in Art. 28 is sufficient to determine maxima and minima if the function and its derivative are one-valued and continuous. In Figs. 31a and 31b are shown some types of maxima and minima which do not satisfy these conditions.

At B and C, Fig. 31a, the tangent is vertical and the derivative infinite. At D the slope on the left is different from that on the right. The derivative is discontinuous. At A and E the curve ends. This happens in problems where values beyond a certain range are impossible. According to our definition, y has maxima at A, B, D and minima at C, E, although the derivative is not zero at any of these points.

If more than one value of the function corresponds to a

single value of the variable, points like A and B, Fig. 31b, may occur. At such points, two values of y coincide.

These diagrams show that in determining maxima and minima special attention must be given to places where the

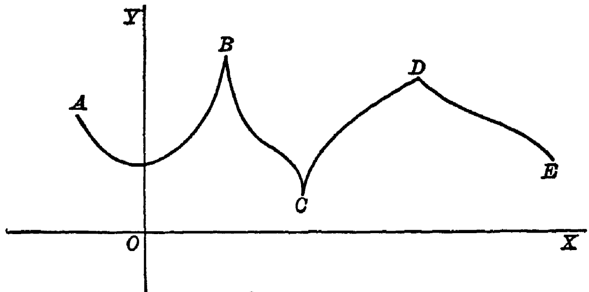

FIG 31a.

derivative is discontinuous, the function ceases to exist, or two values of the function coincide.

Example 1. Find the maximum and minimum ordinates on the curve $y^3 = x^2$.

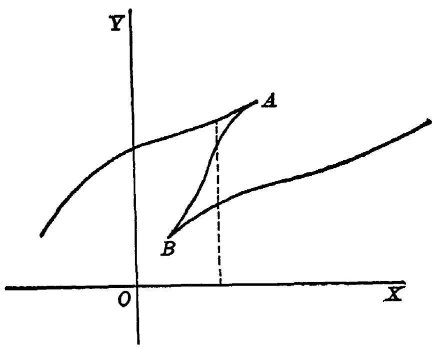

FIG. 31b.

In this case, $y = x^{\frac{2}{3}}$ and

$$\frac{dy}{dx} = \frac{2}{3}\, x^{-\frac{1}{3}}.$$

No finite value of x makes the derivative zero, but $x = 0$ makes it infinite. Since y is never negative, the value 0 is a minimum (Fig. 31c).

Example 2. A man on one side of a river $\frac{1}{2}$ mile wide wishes to reach a point on the opposite side 2 miles down the river. If he can row 6 miles per hour and walk 4, find the route he should take to make the trip in least time.

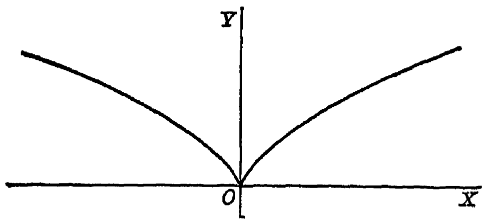

FIG. 31c.

Let A, Fig. 31d, be the starting point and B the destination. Suppose he rows to C, x miles down the river. The total time is then

$$t = \tfrac{1}{6}\sqrt{x^2 + \tfrac{1}{4}} + \tfrac{1}{4}(2 - x).$$

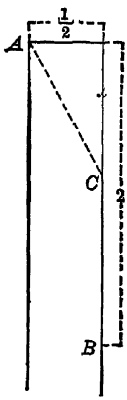

FIG. 31d.

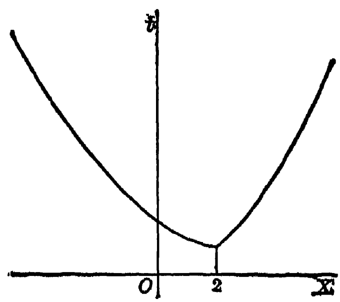

FIG. 31e.

Equating the derivative to zero we get

$$\frac{x}{6\sqrt{x^2 + \frac{1}{4}}} - \tfrac{1}{4} = 0$$

which reduces to

$$5\,x^2 + \tfrac{9}{4} = 0.$$

There is no real solution.

The trouble is that

$$\tfrac{1}{4}(2 - x)$$

is the time of walking only if C is above B. If C is below B, the time is

$$\tfrac{1}{4}(x - 2).$$

The complete value of t is then

$$t = \tfrac{1}{6}\sqrt{x^2 + \tfrac{1}{4}} \pm \tfrac{1}{4}(2 - x),$$

the sign being $+$ if $x<2$ and $-$ if $x>2$. The graph of the equation connecting x and t is shown in Fig. 31e. At $x = 2$ the derivative is discontinuous. Since he rows faster than he walks, the minimum obviously occurs when he rows all the way. That is, $x = 2$.

EXERCISES

1. Find the maximum and minimum values of y on the curve

$$y^3 = x^2(x - 1).$$

2. Find the point on the parabola

$$y^2 = 4x + 4$$

nearest the origin.

3. Find the point on the circle

$$x^2 + y^2 = 1$$

at greatest distance from (1, 0).

4. A wire of length L is cut into two pieces, one of which is bent to form a circle, the other a square. Find the lengths of the two pieces when the sum of the areas of the square and circle is greatest.

CHAPTER VI

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

32. Circular Measure of an Angle. — In work that requires calculus angles are generally expressed in circular measure. Let AOB (Fig. 32) be a given angle. Describe a circular arc $AB = s$ of radius r and center at the vertex of the angle. The circular measure of the angle is defined as the ratio

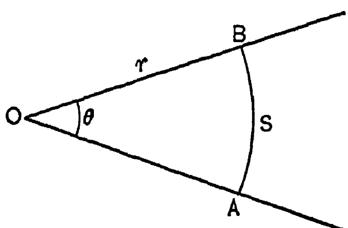

Fig. 32.

$$\theta = \frac{\text{arc}}{\text{rad}} = \frac{s}{r}. \qquad (32a)$$

If the radius and angle are given the arc length is determined by the equation

$$s = r\theta. \qquad (32b)$$

The angle with circular measure unity is called a *radian.* An angle of circular measure θ is then an angle of θ radians.

33. The Sine of a Small Angle. — Inspection of a table of natural sines will show that the sine of a small angle is very nearly equal to the circular measure of the angle. Thus

$$\sin 1^\circ = 0.017452,$$

$$\frac{\pi}{180} = 0.017453.$$

We should then expect that

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1. \qquad (33)$$

To show this graphically, let $\theta = AOP$ (Fig. 33). Draw PM perpendicular to OA and continue this line to Q. The circular measure of the angle is

$$\theta = \frac{\text{arc } AP}{OP}.$$

Also

$$\sin \theta = \frac{MP}{OP}.$$

Hence

$$\frac{\sin \theta}{\theta} = \frac{MP}{\text{arc } AP} = \frac{\text{chord } QP}{\text{arc } QP}.$$

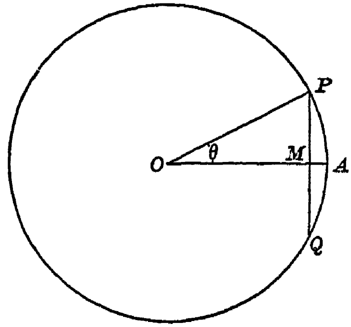

FIG. 33.

As θ approaches zero, the arc QP approaches zero and the ratio of the arc and chord approaches unity (Art. 56). Therefore

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1. \tag{33}$$

34. Formulas for Differentiating Trigonometric Functions. — Let u be the circular measure of an angle.

VII. $\dfrac{d}{dx} \sin u = \cos u \dfrac{du}{dx}.$

VIII. $\dfrac{d}{dx} \cos u = -\sin u \dfrac{du}{dx}.$

IX. $\dfrac{d}{dx} \tan u = \sec^2 u \dfrac{du}{dx}.$

X. $\dfrac{d}{dx} \cot u = -\csc^2 u \dfrac{du}{dx}.$

XI. $\dfrac{d}{dx} \sec u = \sec u \tan u \dfrac{du}{dx}.$

XII. $\dfrac{d}{dx} \csc u = -\csc u \cot u \dfrac{du}{dx}.$

The negative sign occurs in the derivatives of all co-functions.

35. Proof of VII, the Derivative of the Sine. — Let

$$y = \sin u.$$

Then

$$y + \Delta y = \sin (u + \Delta u)$$

and so

$$\Delta y = \sin (u + \Delta u) - \sin u.$$

It is shown in trigonometry that

$$\sin A - \sin B = 2 \cos \tfrac{1}{2} (A + B) \sin \tfrac{1}{2} (A - B).$$

If then $A = u + \Delta u$, $B = u$,

$$\Delta y = 2 \cos (u + \tfrac{1}{2} \Delta u) \sin \tfrac{1}{2} \Delta u,$$

whence

$$\frac{\Delta y}{\Delta x} = 2 \cos (u + \tfrac{1}{2} \Delta u) \frac{\sin \frac{1}{2} \Delta u}{\Delta x}$$
$$= \cos (u + \tfrac{1}{2} \Delta u) \frac{\sin \frac{1}{2} \Delta u}{\frac{1}{2} \Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

As Δu approaches zero

$$\frac{\sin \frac{1}{2} \Delta u}{\frac{1}{2} \Delta u} = \frac{\sin \theta}{\theta}$$

approaches unity and $\cos (u + \frac{1}{2} \Delta u)$ approaches $\cos u$. Therefore in the limit

$$\frac{dy}{dx} = \cos u \frac{du}{dx}.$$

36. Proof of VIII, the Derivative of the Cosine. — By trigonometry

$$\cos u = \sin\left(\frac{\pi}{2} - u\right).$$

Using the formula for the derivative of a sine we then have

$$\frac{d}{dx}\cos u = \frac{d}{dx}\sin\left(\frac{\pi}{2} - u\right) = \cos\left(\frac{\pi}{2} - u\right)\frac{d}{dx}\left(\frac{\pi}{2} - u\right)$$
$$= -\sin u\frac{du}{dx}.$$

37. Proof of IX, X, XI, and XII. — Differentiating both sides of the equation

$$\tan u = \frac{\sin u}{\cos u}$$

and using the formulas just proved for the derivatives of $\sin u$ and $\cos u$,

$$\frac{d}{dx}\tan u = \frac{\cos u\frac{d}{dx}\sin u - \sin u\frac{d}{dx}\cos u}{\cos^2 u}$$
$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\frac{du}{dx} = \sec^2 u\frac{du}{dx}.$$

By differentiating both sides of the equations

$$\cot u = \frac{\cos u}{\sin u}, \quad \sec u = \frac{1}{\cos u}, \quad \csc u = \frac{1}{\sin u}$$

and using the formulas for the derivatives of $\sin u$ and $\cos u$ we obtain formulas X, XI, XII.

Example 1. $y = 3\cos 2x$.

Using formula VIII,

$$\frac{dy}{dx} = -3\sin 2x\frac{d}{dx}(2x) = -6\sin 2x.$$

Example 2. $y = \sin^2(x^2 + 3)$.

Since

$$\sin^2(x^2+3) = [\sin(x^2+3)]^2,$$

we first use the formula for differentiating u^2 and so get

$$\begin{aligned}\frac{dy}{dx} &= 2 \sin (x^2 + 3) \frac{d}{dx} \sin (x^2 + 3) \\ &= 2 \sin (x^2 + 3) \cos (x^2 + 3) \frac{d}{dx} (x^2 + 3) \\ &= 4\, x \sin (x^2 + 3) \cos (x^2 + 3).\end{aligned}$$

Example 3. $y = \frac{1}{2} \sec^4 x - \sec^2 x.$
Differentiating and reducing, we find

$$\begin{aligned}\frac{dy}{dx} &= 2 \sec^3 x \cdot \sec x \tan x - 2 \sec x \cdot \sec x \tan x \\ &= 2 \sec^2 x \tan x (\sec^2 x - 1) \\ &= 2 \sec^2 x \tan x \cdot \tan^2 x \\ &= 2 \sec^2 x \tan^3 x.\end{aligned}$$

EXERCISES

Find $\frac{dy}{dx}$ in each of the following exercises:

1. $y = 4 \sin 3\, x.$
2. $y = 2 \cos \dfrac{x}{2}.$
3. $y = 2 \sin^2 \dfrac{x}{2}.$
4. $y = \frac{1}{3} \cos^3 x.$
5. $y = \dfrac{x}{2} - \frac{1}{4} \sin 2\, x.$
6. $y = \sin 5\, x - \frac{1}{3} \sin^3 5\, x.$
7. $y = \sec^2 x - \tan^2 x.$
8. $y = \cos^3 \dfrac{x}{3} - 3 \cos \dfrac{x}{3}.$
9. $y = \tan 2\, x + \frac{1}{3} \tan^3 2\, x.$
10. $y = \frac{1}{5} \sec^5 x - \frac{2}{3} \sec^3 x + \sec x.$
11. $y = \frac{1}{5} \cot^5 x - \frac{1}{3} \cot^3 x + \cot x + x.$
12. $y = \tan^{\frac{3}{2}} x - 3 \tan^{\frac{1}{2}} x.$
13. $y = x(\frac{1}{3} \cos^3 x - \cos x) + \frac{1}{9} \sin^3 x + \frac{2}{3} \sin x.$
14. $y = \frac{1}{2} \cos x(\frac{1}{3} \sin^5 x + \frac{5}{12} \sin^3 x + \frac{5}{8} \sin x) - \frac{5}{16} x.$
15. $y = \dfrac{1 + \sin \frac{x}{2}}{1 - \sin \frac{x}{2}}.$
16. $y = \dfrac{\sec x + \tan x}{\sec x - \tan x}.$
17. If $y = A \cos nx + B \sin nx$, where A, B, n are constants, show that

$$\frac{d^2y}{dx^2} + n^2 y = 0.$$

18. Find the constant A such that $y = A \sin 2x$ satisfies the equation

$$\frac{d^2y}{dx^2} + 3y = 3 \sin 2x.$$

19. Find A and B such that $y = A \sin 3x + B \cos 3x$ satisfies the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 10 \cos 3x.$$

20. Find the angle at which the curve

$$y = \tfrac{1}{3} \sin 3x$$

crosses the x-axis.

21. Find the maximum height of the curve

$$y = 3 \cos x + 4 \sin x$$

above the x-axis.

22. A weight supported by a spring hangs at rest at the origin. If the weight is lifted the distance a and let fall, its height at any subsequent time t will be

$$y = a \cos kt,$$

k being constant. Find its velocity and acceleration as it passes the origin. Where is the velocity greatest? Where is the acceleration greatest?

23. A point P moves with constant speed around a circle of radius a and center O Show that its projection M on a fixed diameter of the circle has an acceleration proportional to OM.

24. A revolving light 5 miles from a straight shore line makes two complete revolutions per minute. Find the velocity along the shore of the beam of light at the instant when it makes an angle of 60° with the shore line.

25. At a certain instant the sides of a right triangle are $AB = 3$, $BC = 4$, $AC = 5$. If AB is constant and the angle BAC increasing 1° per second, find the rate of increase of the area at that instant.

26. Given that two sides and the included angle of a triangle have at a certain instant the values 6 ft , 10 ft., and 30° respectively, and that these quantities are changing at the rates of 2 ft., −3 ft., and 10° per second, how fast is the area of the triangle then changing?

27. OA is a crank revolving with angular velocity ω about O. AB is a connecting rod attached to a piston B which moves along a fixed line through O. Show that the speed of B is $\omega \cdot OC$, where C is the point in which the line BA intersects the line through O perpendicular to OB.

28. A triangle has two equal sides of length a. Find the angle between these sides if the area is a maximum.

29. An alley 8 ft. wide runs perpendicular to a street 27 ft. wide. What is the longest beam that can be moved horizontally along the street into the alley?

30. A needle rests with one end in a smooth hemispherical bowl. The needle will sink to a position in which its center is as low as possible. If the length of the needle equals the diameter of the bowl, what will be the position of equilibrium?

31. A rope with a ring at one end is looped over two pegs in a horizontal line and held taut by a weight fastened to the free end. If the rope slips freely, the weight will descend as far as possible. Find the angle formed at the bottom of the loop.

32. Find the angle at the bottom of the loop in the preceding problem if the rope is looped over a circular pulley instead of the two pegs.

33. A gutter is made by bending into shape a strip of copper so that the cross section is an arc of a circle. If the width of the strip is a, find the radius of the circle when the carrying capacity is greatest

34. The lower corner of a page is folded over so as just to reach the inner edge. If the width of the page is 6 inches, find the minimum length of crease

35. A vertical telegraph pole at a bend in the line is supported from tipping over by a stay 40 ft. long fastened to the pole and to a stake in the ground. How far from the pole should the stake be driven to make the tension in the stay as small as possible? The moment of the tension (Art. 103) about the foot of the pole will be constant.

38. Inverse Trigonometric Functions. — The symbol $\sin^{-1} x$ is used to represent the angle whose sine is x. Thus

$$y = \sin^{-1} x, \qquad x = \sin y$$

are equivalent equations. Similar definitions apply to $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$.

Since supplementary angles and those differing by 2π have the same sine, an infinite number of angles are represented by the same symbol $\sin^{-1} x$. The algebraic sign of the derivative depends on the angle differentiated. In the formulas given below it is assumed that $\sin^{-1} u$ and $\csc^{-1} u$ are angles in the first or fourth quadrant, $\cos^{-1} u$ and $\sec^{-1} u$ angles in the first or second quadrant. If angles in other

quadrants are differentiated, the opposite sign must be used. The formulas for $\tan^{-1} u$ and $\cot^{-1} u$ are valid in all quadrants.

39. Formulas for Differentiating Inverse Trigonometric Functions. —

$$\text{XIII.} \quad \frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}.$$

$$\text{XIV.} \quad \frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}.$$

$$\text{XV.} \quad \frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}.$$

$$\text{XVI.} \quad \frac{d}{dx}\cot^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}.$$

$$\text{XVII.} \quad \frac{d}{dx}\sec^{-1} u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}.$$

$$\text{XVIII.} \quad \frac{d}{dx}\csc^{-1} u = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}.$$

40. Proof of the Formulas. — Let

$$y = \sin^{-1} u.$$

Then

$$\sin y = u.$$

Differentiating with respect to x,

$$\cos y \frac{dy}{dx} = \frac{du}{dx},$$

whence

$$\frac{dy}{dx} = \frac{1}{\cos y}\frac{du}{dx}.$$

If y is an angle in the first or fourth quadrants, $\cos y$ is positive and

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1-u^2},$$

hence

$$\frac{dy}{dx} = \frac{1}{\cos y}\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}.$$

The other formulas may be proved in a similar way.

Example. $y = \sec^{-1} \frac{1}{2}\left(x + \frac{1}{x}\right).$

By formula XVII

$$\frac{dy}{dx} = \frac{\frac{1}{2}\left(1 - \frac{1}{x^2}\right)}{\frac{1}{2}\left(x + \frac{1}{x}\right)\sqrt{\frac{1}{4}\left(x + \frac{1}{x}\right)^2 - 1}} = \frac{2}{x^2+1}.$$

EXERCISES

Find $\frac{dy}{dx}$ in each of the following exercises:

1. $y = \sin^{-1}(3x - 1).$
2. $y = \cos^{-1}\left(1 - \frac{x}{a}\right).$
3. $y = \tan^{-1} \frac{3}{2} x.$
4. $y = \cot^{-1} \frac{1}{2}\left(x - \frac{1}{x}\right).$
5. $y = \sec^{-1}\sqrt{4x+1}.$
6. $y = \frac{1}{2}\csc^{-1}\frac{3}{4x-1}.$
7. $y = \tan^{-1}\frac{x-a}{x+a}.$
8. $y = \csc^{-1}(\sec x).$
9. $y = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$
10. $y = a\sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}.$
11. $y = \sqrt{x^2 - a^2} - a\sec^{-1}\frac{x}{a}.$
12. $y = \frac{2x}{x^2+4} + \tan^{-1}\frac{x}{2}.$
13. $y = x\cos^{-1} 2x - \frac{1}{2}\sqrt{1 - 4x^2}.$
14. $y = x\sin^{-1} x + \sqrt{1 - x^2}.$
15. $y = \sqrt{x^2 - 4} - 2\tan^{-1}\frac{\sqrt{x^2 - 4}}{2}.$
16. $y = \frac{\sqrt{x^2 - a^2}}{x^2} - \frac{1}{a}\csc^{-1}\frac{x}{a}.$
17. $y = \tan^{-1}\frac{4\sin x}{3 + 5\cos x}.$

18. $y = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)$.

19. $y = \dfrac{2}{\sqrt{4\,ac - b^2}}\tan^{-1}\dfrac{2\,ax + b}{\sqrt{4\,ac - b^2}}$.

20. $y = \sec^{-1}\dfrac{x+1}{x-1} + \sin^{-1}\dfrac{x-1}{x+1}$

21. If s is the arc from the x-axis to the point (x, y) on the circle $x^2 + y^2 = a^2$, show that

$$s = a\cos^{-1}\frac{x}{a}, \qquad \frac{ds}{dx} = -\frac{a}{y}.$$

22. If A is the area bounded by the circle $x^2 + y^2 = a^2$, the y-axis and the vertical through (x, y) show that

$$A = xy + a^2\sin^{-1}\frac{x}{a}, \qquad \frac{dA}{dx} = 2\,y.$$

23. A tablet 7 ft. high is placed on a wall with its base 9 ft. above the level of an observer's eye. How far from the wall should the observer stand that the angle of vision subtended by the tablet may be a maximum?

24. The end of a string wound on a pulley of radius a moves with velocity v along a line perpendicular to the axis of the pulley. If the string is kept taut, find the angular velocity with which the pulley turns when the end of the string is at distance x from the center of the pulley.

41. Exponential and Logarithmic Functions. — If a is a positive constant, a^u is called an exponential function. If u is a fraction, it is understood that a^u is the positive root.

If $y = a^u$, then u is called the logarithm of y to base a. That is,

$$y = a^u, \qquad u = \log_a y$$

are by definition equivalent equations. Elimination of u gives the important identity

$$a^{\log_a y} = y. \tag{41}$$

This expresses symbolically that the logarithm of y is the power to which the base must be raised to equal y.

42. Definition of e. — In proving the formula for the derivative of a logarithm we shall find it necessary to determine the limit approached by the expression

$$(1+h)^{\frac{1}{h}}$$

when h approaches zero.

From the binomial theorem we have

$$(1+h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \frac{n(n-1)(n-2)}{3!}h^3 + \cdots.$$

In algebra this is proved only when n is a positive integer. We shall see later (Chap. XIX) that the same formula is valid for negative and fractional values of n provided h is in numerical value less than 1. If then we take

$$n = \frac{1}{h},$$

we have

$$(1+h)^{\frac{1}{h}} = 1 + \frac{1}{h}h + \frac{\frac{1}{h}\left(\frac{1}{h}-1\right)}{2!}h^2 + \frac{\frac{1}{h}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{3!}h^3 + \cdots$$

$$= 1 + 1 + \frac{1}{2!}(1-h) + \frac{1}{3!}(1-h)(1-2h) + \cdots.$$

When h approaches zero, this approaches the limit

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = 2.7183$$

approximately. Hence

$$\lim_{h\to 0}(1+h)^{\frac{1}{h}} = e. \tag{42}$$

Logarithms to base e are called *natural logarithms.* In this book we shall represent these by the symbol ln. Thus $\ln u$ means the natural logarithm of u, or logarithm to base e.

43. Derivatives of Exponential and Logarithmic Functions. —

XIX. $\dfrac{d}{dx}\log_a u = \dfrac{\log_a e}{u}\dfrac{du}{dx}.$

XX. $\dfrac{d}{dx}\ln u = \dfrac{1}{u}\dfrac{du}{dx}.$

XXI. $\dfrac{d}{dx}a^u = a^u \ln a\dfrac{du}{dx}.$

XXII. $\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}.$

44. Proof of XIX and XX, the Derivative of a Logarithm.— Let

$$y = \log_a u,$$

where u is a function of x. Then

$$y + \Delta y = \log_a (u + \Delta u)$$

and so

$$\Delta y = \log_a(u+\Delta u) - \log_a u = \log_a \frac{u+\Delta u}{u} = \log_a\left(1+\frac{\Delta u}{u}\right).$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{\log_a\left(1+\dfrac{\Delta u}{u}\right)}{\Delta x} = \frac{\log_a\left(1+\dfrac{\Delta u}{u}\right)}{\dfrac{\Delta u}{u}}\cdot\frac{1}{u}\frac{\Delta u}{\Delta x}. \tag{44a}$$

Let

$$\frac{\Delta u}{u} = h.$$

As Δx approaches zero, Δu approaches zero and so h approaches zero. Hence, by (42),

$$\frac{\log_a\left(1+\dfrac{\Delta u}{u}\right)}{\dfrac{\Delta u}{u}} = \frac{\log_a(1+h)}{h} = \log_a(1+h)^{\frac{1}{h}}$$

approaches $\log_a e$. Therefore (44a) gives in the limit

$$\frac{dy}{dx} = \log_a e \cdot \frac{1}{u}\frac{du}{dx}, \tag{44b}$$

proving formula XIX.

Equation (44b) is valid for any value of the positive constant a. In particular, if $a = e$

$$\log_a e = \log_e e = 1$$

and

$$\frac{dy}{dx} = \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

which is formula XX.

45. Proof of XXI and XXII, the Derivative of an Exponential. — Let

$$y = a^u.$$

Taking natural logarithms of both sides,

$$\ln y = u \ln a.$$

Differentiating by formula XX,

$$\frac{1}{y}\frac{dy}{dx} = \ln a \frac{du}{dx},$$

whence

$$\frac{dy}{dx} = y \ln a \frac{du}{dx} = a^u \ln a \frac{du}{dx}. \tag{45a}$$

This proves formula XXI. In particular, if $a = e$,

$$\ln a = \ln e = 1$$

and (45a) becomes

$$\frac{dy}{dx} = \frac{d}{dx} e^u = e^u \frac{du}{dx}$$

which is formula XXII.

Example 1. $y = \ln \sec^2 x$.

By formula XX,

$$\frac{dy}{dx} = \frac{\dfrac{d}{dx}\sec^2 x}{\sec^2 x} = \frac{2 \sec x\ (\sec x \tan x)}{\sec^2 x} = 2 \tan x.$$

Example 2. $y = a^{\tan^{-1} x}$.

$$\frac{dy}{dx} = a^{\tan^{-1} x} \ln a \frac{d}{dx} \tan^{-1} x = \frac{a^{\tan^{-1} x} \ln a}{1 + x^2}.$$

EXERCISES

Find $\dfrac{dy}{dx}$ in each of the following exercises:

1. $y = \ln (3x^2 + 5x + 1)$.
2. $y = \ln (x^2 - 4x + 4)$.
3. $y = \ln \tan x$.
4. $y = x \ln x - x$.
5. $y = \ln \dfrac{x-1}{x+1}$.
6. $y = \log_{10} (x^2 - 2x)$. ✓
7. $y = \ln (\sec x + \tan x)$.
8. $y = \ln (x + \sqrt{x^2 - a^2})$.
9. $y = \ln \sin x + \frac{1}{2} \cos^2 x$.
10. $y = \ln (\sqrt{x+3} + \sqrt{x+2})$.
11. $y = \frac{1}{3} \ln \tan \frac{3}{2} x$.
12. $y = \frac{1}{2} (e^x + e^{-x})$.
13. $y = x^2 + 2^x$.
14. $y = x^e + e^x$.
15. $y = x^n n^x$.
16. $y = e^{\sin x}$.
17. $y = (3x + 1) e^{-3x}$.
18. $y = \dfrac{x}{2} e^{2x} - \frac{1}{4} e^{2x}$.
19. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.
20. $y = (x^2 - 2x + 2)e^x$.
21. $y = \frac{1}{5} e^x (\sin 2x - 2 \cos 2x)$.
22. $y = \frac{1}{4} [x - \ln (4 + e^x)]$.
23. $y = x \sec^{-1} x - \ln (x + \sqrt{x^2 - 1})$.
24. $y = x \tan^{-1} \dfrac{x}{2} - \ln (x^2 + 4)$.

25. $y = \frac{1}{4} \ln \dfrac{x^2}{x^2 - 4} - \dfrac{1}{x^2 - 4}$.

26. $y = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln \cos x$.

27. $y = \dfrac{1}{a}\left[\cos ax + \ln \tan \dfrac{ax}{2}\right]$.

28. $y = \ln (\ln ax)$.

29. $y = \dfrac{x}{2} [\sin (\ln x) - \cos (\ln x)]$.

30. $y = \dfrac{x}{8} (2x^2 + 5) \sqrt{x^2 + 1} + \frac{3}{8} \ln (x + \sqrt{x^2 + 1})$.

31. Find the slope of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

at the point $x = a$.

32. Find the points on the curve $y = e^x \sin x$ where the tangent is parallel to the x-axis.

33. Find the maximum value of the function $x^2e^{-x^2}$.

34. If $y = Ae^{nx} + Be^{-nx}$ where A, B, and n are constants, show that

$$\frac{d^2y}{dx^2} - n^2y = 0.$$

35. If $y = ze^{-3x}$, where z is a function of x, show that

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{-3x}\frac{d^2z}{dx^2}.$$

36. If a particle moves along a straight line the distance

$$s = ae^{kt} + be^{-kt}$$

in time t, show that its acceleration is proportional to s.

37. If a particle falls the distance

$$s = \frac{g}{k^2} \ln \left(\frac{e^{kt} + e^{-kt}}{2}\right)$$

in time t, show that its velocity v and acceleration a satisfy the equation

$$a = g - \frac{k^2v^2}{g}.$$

CHAPTER VII

DIFFERENTIALS

46. Infinitesimals. — A variable approaching zero as limit is called an *infinitesimal.* Such for example are the increments Δx and Δy in the limit process by which the derivative $\frac{dy}{dx}$ is determined.

In case several infinitesimals are functions of a single one considered as independent variable, that one is called the *principal* infinitesimal. In most applications of calculus the increment of the independent variable is taken as principal infinitesimal and other infinitesimals are considered as functions of it.

Two numbers are ordinarily considered approximately equal when their difference is small. In case of numbers approaching zero that method of comparison is not satisfactory since it makes all such numbers approximately equal. Consequently small numbers are compared by ratio and not by difference, two such numbers being considered approximately equal when their ratio is approximately unity.

Two infinitesimals α and β, which are functions of the same principal infinitesimal, are said to be of the same *order* if the ratio

$$\frac{\alpha}{\beta}$$

approaches a limit finite and different from zero when the principal infinitesimal approaches zero. If

$$\lim \frac{\alpha}{\beta} = 0,$$

α is said to be of higher order than β and if this ratio becomes infinite α is of lower order than β.

Thus, if Δx is infinitesimal and $y = f(x)$ is a continuous function, Δy approaches zero when Δx approaches zero and so Δy is also infinitesimal. At a point where the derivative has a definite value

$$\lim \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

different from zero, Δy and Δx are infinitesimals of the same order. At a point where

$$\frac{dy}{dx} = 0,$$

Δy is of higher order than Δx and at a point where

$$\frac{\Delta y}{\Delta x}$$

becomes infinite, Δy is of lower order than Δx.

If α is the principal infinitesimal, an infinitesimal of the same order as α is said to be of the first order and an infinitesimal of the same order as α^n is said to be of the nth order. Thus, if Δx is the principal infinitesimal, $(\Delta x)^2$ is one of the second order, $(\Delta x)^3$ of the third order, etc.

Example. If α is the principal infinitesimal, show that $\sin^2 \alpha$ is an infinitesimal of the second order.

Since

$$\lim_{\alpha \to 0} \frac{\sin^2\alpha}{\alpha^2} = \lim\left(\frac{\sin \alpha}{\alpha}\right)^2 = 1,$$

$\sin^2 \alpha$ is an infinitesimal of the same order as α^2, that is, of the second order.

47. Approximate Value of the Increment of a Function. — Let $y = f(x)$ be a function which for each value of x considered has a finite derivative

$$\frac{dy}{dx} = f'(x).$$

When x receives an increment Δx, an increment Δy is determined. When Δx approaches zero, we shall now show that Δy becomes approximately equal to

$$f'(x)\ \Delta x,$$

the error being an infinitesimal of higher order than Δx.

To show this let ϵ be the difference of Δy and $f'(x)\ \Delta x$. That is, let

$$\Delta y = f'(x)\ \Delta x + \epsilon.$$

Then

$$\frac{\Delta y}{\Delta x} = f'(x) + \frac{\epsilon}{\Delta x}.$$

When Δx approaches zero, by the definition of derivative

$$\frac{\Delta y}{\Delta x}$$

approaches $f'(x)$ and so

$$\frac{\epsilon}{\Delta x}$$

approaches zero. Hence ϵ is an infinitesimal of higher order than Δx.

If then we take $f'(x)\ \Delta x$ as an approximate value for Δy, by making Δx sufficiently small, we can make the error as small as we please *in comparison with* Δx.

Example. Find the approximate change in $y = x^2$ when x changes from 2 to 2.1 and from 2 to 2.01, comparing the error with Δx in each case.

The derivative is

$$f'(x) = \frac{dy}{dx} = 2\,x.$$

When x changes from 2 to 2.1, the approximate change in y is

$$f'(x)\ \Delta x = 2\,x\ \Delta x = 2\ (2)\ (0.1) = 0.40.$$

The exact value is

$$\Delta y = (2.1)^2 - 2^2 = 0.41.$$

Hence the error in the above approximation is

$$.01 = \tfrac{1}{10}\,\Delta x.$$

Similarly, when x changes from 2 to 2.01, the approximate increment is

$$f'(x)\ \Delta x = 2\,(2)\,(.01) = .04$$

and the exact increment is

$$\Delta y = (2.01)^2 - 2^2 = .0401.$$

The error in this case is

$$.0001 = \tfrac{1}{100}\,\Delta x.$$

As Δx decreases the error in the approximation becomes a smaller and smaller fraction of Δx.

48. Differential of a Function of a Single Independent Variable. — If x is the independent variable and

$$y = f\,(x),$$

the quantity $f'(x)\ \Delta x$ (which for small values of Δx we have just shown to be approximately equal to Δy) is called the *differential* of y and represented by the symbol dy. That is,

$$dy = f'(x)\ \Delta x. \qquad (48a)$$

In particular, if $f(x) = x$, this equation becomes

$$dx = \Delta x. \qquad (48b)$$

That is, *the differential of the independent variable is equal to its increment and the differential of any function is equal to the product of its derivative and the increment of the independent variable.*

Substituting dx for Δx in (48a), we have

$$dy = f'(x)\,dx, \tag{48c}$$

whence

$$\frac{dy}{dx} = f'(x).$$

That is, *the quotient dy divided by dx is equal to the derivative of y with respect to x.*

In the preceding chapters we have considered the symbol

$$\frac{dy}{dx}$$

as merely a notation for the derivative. If x is the independent variable we have just seen that the derivative may be regarded as a fraction with numerator dy and denominator dx. We shall now show that (48c) is satisfied whether x is the independent variable or not. Hence the derivative of y with respect to x is equal to the ratio of dy to dx whenever dx is not zero.

To show this let t be the independent variable and $y = f(x)$,

$$x = \phi(t), \qquad y = \psi(t).$$

By the definition of differentials,

$$dx = \phi'(t)\,\Delta t, \qquad dy = \psi'(t)\,\Delta t,$$

where $\phi'(t)$ and $\psi'(t)$ are the derivatives of $\phi(t)$ and $\psi(t)$ with respect to t. When t receives the increment Δt, let the corresponding increments in x and y be Δx and Δy. Then from algebra

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}.$$

Assuming that $\psi'(t)$, $f'(x)$, $\phi'(t)$ are all finite and definite, when Δt approaches zero this equation gives in the limit

$$\psi'(t) = f'(x)\,\phi'(t).$$

Multiplying both sides by Δt, we then have

$$\psi'(t)\ \Delta t = f'(x) \cdot \phi'(t)\ \Delta t$$

or

$$dy = f'(x)\ dx$$

which was to be proved.

Example. If t is the independent variable and

$$x = t^2, \qquad y = t^3 + t,$$

determine the derivative of y with respect to x and by direct comparison show that it is equal to the ratio of dy to dx.

Eliminating t, we get

$$y = x^{\frac{3}{2}} + x^{\frac{1}{2}}.$$

From this the derivative of y with respect to x is found to be

$$\tfrac{3}{2}\, x^{\frac{1}{2}} + \tfrac{1}{2}\, x^{-\frac{1}{2}},$$

which becomes

$$\frac{3}{2} t + \frac{1}{2\,t}$$

when $x^{\frac{1}{2}}$ is replaced by its value t.

Since t is the independent variable, by direct definition of differentials,

$$dx = 2\,t\ \Delta t, \qquad dy = (3\,t^2 + 1)\ \Delta t,$$

whence the ratio of dy to dx is

$$\frac{dy}{dx} = \frac{3\,t^2 + 1}{2\,t} = \frac{3}{2} t + \frac{1}{2\,t}$$

which is identical with the value found above for the derivative.

49. Graphical Representation of Differential and Increment. — Let the curve in Fig. 49a represent the graph of the equation $y = f(x)$. The slope of the curve at $P(x, y)$ is

$$f'(x) = \tan (RPT).$$

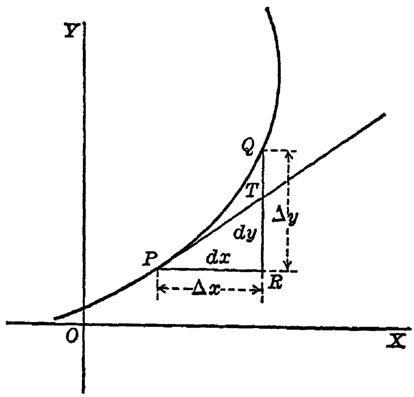

FIG. 49a.

In passing from P to Q the increments of x and y are

$$\Delta x = PR, \qquad \Delta y = RQ.$$

If x is taken as independent variable,

$$dx = \Delta x = PR,$$
$$dy = f'(x)\,\Delta x = PR \tan (RPT) = RT.$$

Thus Δx and Δy are the increments in x and y when we move from the point P to a neighboring point Q on the curve, whereas dx and dy are the increments when we move from P to a point T on the tangent to the curve at P. Since dx and dy are the sides of the triangle PRT their ratio is obviously equal to the slope of the tangent at P.

A point, describing the curve, when passing through P is moving in the direction of the tangent PT. *The differential dy is thus the amount y would change while x is changing the amount Δx if the point continued to move in the same direction as at P.* In general it does not continue in the same direc-

tion and so dy and Δy are different. But if the arc is sufficiently short the change in direction is usually small and so dy and Δy are nearly equal. In fact, from Art. 47, if there is a definite slope at each point,

$$\Delta y - dy = TQ$$

is an infinitesimal of higher order than Δx.

Similarly, if t is the independent variable and s the distance a body moves in time t, the equation

$$ds = \frac{ds}{dt}dt = v\ \Delta t$$

expresses that *ds is the distance the body would move in the time Δt if it continued with the speed it has at the time t.* In general the speed does not remain constant and so Δs is different from ds, but if the interval is sufficiently short the speed is nearly constant and these quantities are nearly equal.

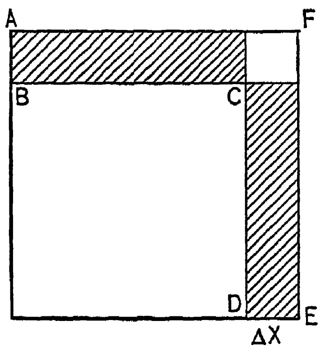

FIG. 49b.

Example. If A is the area of a square of side x, represent dA and ΔA on a diagram and show that $\Delta A - dA$ is an infinitesimal of higher order than Δx.

In Fig. 49b, when x receives the increment Δx the increment in the area consists of three rectangles,

$$\Delta A = AC + CE + CF,$$

whereas the differential is

$$dA = 2\,x\,\Delta x = AC + CE.$$

The difference is the corner square

$$\Delta A - dA = CF = (\Delta x)^2$$

whose area is an infinitesimal of higher order than Δx.

We may think of the area as growing through the motion of the lines BC and CD sidewise at constant speed. When the side of the square is x, area is being formed along the lines BC, CD. To obtain the differential this growth must be continued at a constant rate. The lines must then move without change of length and so sweep out the rectangles AC, CE whose sum consequently represents dA.

EXERCISES

1. If x is infinitesimal, show that $\sin x$ and $\tan x$ are infinitesimals of the same order as x.

2. If x is infinitesimal, show that $x - \sin x$ is an infinitesimal of higher order than x.

3. If θ is the principal infinitesimal, by using the trigonometric identity

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2},$$

show that $1 - \cos\theta$ is an infinitesimal of the second order.

4. If x is the principal infinitesimal, show that

$$2 - x - 2\sqrt{1 - x}$$

is an infinitesimal of the second order.

5. If $y = x^4$ and Δx is sufficiently small, show that

$$\Delta y = 4\,x^3\,\Delta x$$

approximately. Express the error as a fraction of Δx when

$$x = 1, \qquad \Delta x = 0.1$$

and when

$$x = 1, \qquad \Delta x = 0.01.$$

6. If $y = \sin\theta$ and $\Delta\theta$ (expressed in radians) is sufficiently small, show that

$$\Delta y = \cos\theta\,\Delta\theta$$

approximately. Express the error as a fraction of $\Delta\theta$ when

$$\theta = 10^\circ, \qquad \Delta\theta = 1^\circ$$

and when

$$\theta = 10^\circ, \qquad \Delta\theta = 1'.$$

7. Show that

$$\tan 46^\circ - \tan 45^\circ = \frac{\pi}{90}$$

approximately. Express the error as a fraction of 1° in radians.

8. If x is the independent variable and

$$y = \sqrt{x^2 + 9},$$

find dy and Δy when x changes from 4 to 4.01 and express the difference

$$\Delta y - dy$$

as a fraction of Δx.

9. If θ is the independent variable and

$$y = \cos \theta$$

find dy and Δy when θ changes from 45° to 46° and express the difference $\Delta y - dy$ as a fraction of $\Delta\theta$.

10. If $y = x^2$, $x = t^3$ and t is the independent variable, find dx and dy in terms of t and Δt and show that they satisfy the equation

$$dy = 2\,x\,dx.$$

11. If t is the independent variable and

$$x = \sqrt{1 - t^2}, \qquad y = \sqrt{1 + t^2},$$

find dy and dx and show that the derivative of y with respect to x is equal to their ratio.

12. If θ is the independent variable and

$$x = \cos \theta, \qquad y = \sin \theta$$

find dx and dy and show that the derivative of y with respect to x is equal to their ratio.

13. If v is the volume of a cube of side x (independent variable), make a diagram showing dv and Δv and from this prove that $\Delta v - dv$ is an infinitesimal of higher order than Δx.

14. If A is the area of a circle of radius r (independent variable) show that dA is equal to the area of a rectangle of width Δr and length equal to the circumference of the circle.

15. If v is the volume, S the area of surface, and r the radius of a sphere, show that

$$dv = S\,dr.$$

16. If v is the volume, h the altitude, and r the radius of a variable cylinder, show that

$$dv = \pi r^2\,dh + 2\,\pi r h\,dr.$$

50. Differential Formulas. — The differential of a given function can be found by the use of formulas very similar to those we have employed in finding derivatives.

I. $dc = 0.$

II. $d(u + v) = du + dv.$

III. $d(cu) = c\,du.$

IV. $d(uv) = u\,dv + v\,du.$

V. $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}.$

VI. $du^n = nu^{n-1}\,du.$

VII. $d \sin u = \cos u\,du.$

VIII. $d \cos u = -\sin u\,du.$

IX. $d \tan u = \sec^2 u\,du.$

X. $d \cot u = -\csc^2 u\,du.$

XI. $d \sec u = \sec u \tan u\,du.$

XII. $d \csc u = -\csc u \cot u\,du.$

XIII. $d \sin^{-1} u = \dfrac{du}{\sqrt{1 - u^2}}.$

XIV. $d \cos^{-1} u = -\dfrac{du}{\sqrt{1 - u^2}}.$

XV. $d \tan^{-1} u = \dfrac{du}{1 + u^2}.$

XVI. $d \cot^{-1} u = -\dfrac{du}{1 + u^2}.$

XVII. $d \sec^{-1} u = \dfrac{du}{u\sqrt{u^2 - 1}}.$

XVIII. $d \csc^{-1} u = -\dfrac{du}{u\sqrt{u^2 - 1}}.$

XIX. $d \log_a u = \dfrac{\log_a e\,du}{u}.$

XX. $d \ln u = \dfrac{du}{u}.$

XXI. $da^u = a^u \ln a\,du.$

XXII. $de^u = e^u\,du.$

In these formulas c, n, a, e are constants and u, v differentiable functions of a single independent variable. These formulas are obtained by multiplying both sides of the corresponding derivative formulas by dx.

Example 1. $y = x^3 - 3x^2 + 2$.

By formulas II and VI,

$$dy = 3x^2\,dx - 6x\,dx = (3x^2 - 6x)\,dx.$$

Example 2. $x^2 - xy + y^2 = 1$.

Differentiating term by term, we get

$$2x\,dx - (x\,dy + y\,dx) + 2y\,dy = 0,$$

whence

$$dy = \frac{y - 2x}{2y - x}\,dx.$$

51. Higher Derivatives. — The second derivative

$$\frac{d^2y}{dx^2}$$

can be regarded as the quotient obtained by dividing a second differential

$$d^2y = d(dy)$$

by $(dx)^2$. In that case the value of d^2y will however depend on the variable x with respect to which y is differentiated.

Thus, suppose

$$y = x^2, \qquad x = t^3.$$

If we consider y as a function of x,

$$\frac{d^2y}{dx^2} = 2,$$

whence

$$d^2y = 2\,(dx)^2 = 2\,(3t^2\,dt)^2 = 18\,t^4\,dt^2.$$

If, however, we consider y as a function of t,

$$y = x^2 = t^6$$

we find

$$\frac{d^2y}{dt^2} = 30\,t^4,$$

whence

$$d^2y = 30\,t^4\,dt^2$$

which is not equal to the value obtained before.

For this reason we shall not consider the second derivative as a fraction with numerator d^2y and denominator dx^2. Two derivatives, such as

$$\frac{d^2y}{dx^2}, \qquad \frac{d^2y}{dt^2},$$

must not be combined like fractions since d^2y may not have the same value in the two cases.

Since

$$\frac{d^2y}{dx^2}$$

is the first derivative with respect to x of the function $\frac{dy}{dx}$ it is, however, correct to write it in the form

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx}. \tag{51}$$

and so consider it as a fraction with numerator equal to the differential of $\frac{dy}{dx}$ and denominator dx.

In a similar manner we can express the nth derivative as a fraction of the form

$$\frac{d^ny}{dx^n} = \frac{d\left(\frac{d^{n-1}y}{dx^{n-1}}\right)}{dx}.$$

Example 1. Given

$$x = t - \frac{1}{t}, \qquad y = t + \frac{1}{t}$$

find $\dfrac{d^2y}{dx^2}$.

In this case

$$\frac{dy}{dx} = \frac{dt - \dfrac{dt}{t^2}}{dt + \dfrac{dt}{t^2}} = \frac{t^2 - 1}{t^2 + 1}.$$

Consequently

$$\frac{d^2y}{dx^2} = \frac{d\left(\dfrac{t^2 - 1}{t^2 + 1}\right)}{dx} = \frac{\dfrac{4\,t\,dt}{(t^2 + 1)^2}}{\left(1 + \dfrac{1}{t^2}\right)dt} = \frac{4\,t^3}{(t^2 + 1)^3}.$$

Example 2. If y is a function of x and x a function of t, express

$$\frac{d^2y}{dx^2}$$

in terms of derivatives with respect to t.

By algebra

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}.$$

Hence

$$\frac{d^2y}{dx^2} = \frac{d\left(\dfrac{dy}{dx}\right)}{dx} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3}.$$

EXERCISES

Find the differentials of each of the following functions:

1. $\tan\theta - \theta$.
2. $\sqrt{x^2 - a^2} - a \sec^{-1}\dfrac{x}{a}$.
3. $x^2 + xy - y^2$.
4. $\sqrt{x^2 + y^2}$.
5. $\ln xy$.
6. $\tan^{-1}\dfrac{y}{x}$.
7. $(x + y)e^{x-y}$.
8. ze^{x+y}.

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in each of the following exercises:

9. $x = a\cos t,\ y = b\sin t$.
10. $x = \frac{1}{2}(e^t + e^{-t}),\ y = \frac{1}{2}(e^t - e^{-t})$.
11. $x = a\cos^2\theta,\ y = b\sin^2\theta$.
12. $x = \sec\theta,\ y = \tan\theta$.
13. $x = \csc\theta + \cot\theta,\ y = \csc\theta - \cot\theta$.
14. $x = e^t\cos t,\ y = e^t\sin t$.
15. Given $x = \frac{1}{2}t^2 + t,\ y = \frac{1}{2}t^2 - t$, find $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^2x}{dy^2}$.
16. Given $x = r\cos\theta,\ y = r\sin\theta$, express $\dfrac{dy}{dx}$ in terms of $\dfrac{dr}{d\theta}$.
17. By differentiating both sides of the equation

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

with respect to x, express $\dfrac{d^2y}{dx^2}$ in terms of derivatives of x with respect to y.

CHAPTER VIII

GEOMETRICAL APPLICATIONS

52. Tangent Line and Normal. — Let m_1 be the slope of a given curve at $P_1\ (x_1, y_1)$. It is shown in analytic geometry that a line through (x_1, y_1) with slope m_1 is represented by the equation

$$y - y_1 = m_1 (x - x_1).$$

This equation then represents the tangent at (x_1, y_1) where the slope of the curve is m_1.

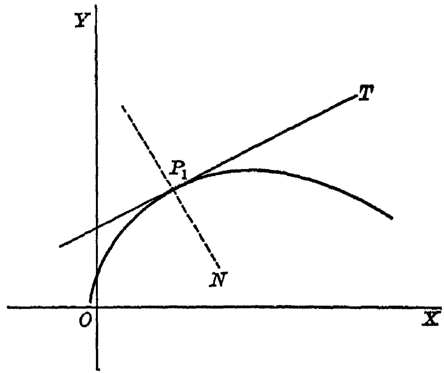

Fig. 52.

The line P_1N perpendicular to the tangent at its point of contact is called the *normal* to the curve at P_1. Since the slope of the tangent is m_1, the slope of a perpendicular line is $-\dfrac{1}{m_1}$ and so

$$y - y_1 = -\frac{1}{m_1}(x - x_1)$$

is the equation of the normal at (x_1, y_1).

Example 1. Find the equations of the tangent and normal to the ellipse $x^2 + 2y^2 = 9$ at the point $(1, 2)$.

The slope at any point of the curve is

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

At $(1, 2)$ the slope is then

$$m_1 = -\tfrac{1}{4}.$$

The equation of the tangent is

$$y - 2 = -\tfrac{1}{4}(x - 1),$$

and the equation of the normal is

$$y - 2 = 4(x - 1).$$

Ex. 2. Find the equation of the tangent to $x^2 - y^2 = a^2$ at the point (x_1, y_1).

The slope at (x_1, y_1) is $\dfrac{x_1}{y_1}$. The equation of the tangent is then

$$y - y_1 = \frac{x_1}{y_1}(x - x_1)$$

which reduces to

$$x_1x - y_1y = x_1^2 - y_1^2.$$

Since (x_1, y_1) is on the curve, $x_1^2 - y_1^2 = a^2$. The equation of the tangent can therefore be reduced to the form

$$x_1x - y_1y = a^2.$$

53. Angle between Two Curves. — By the angle between two curves at a point of intersection we mean the angle between their tangents at that point.

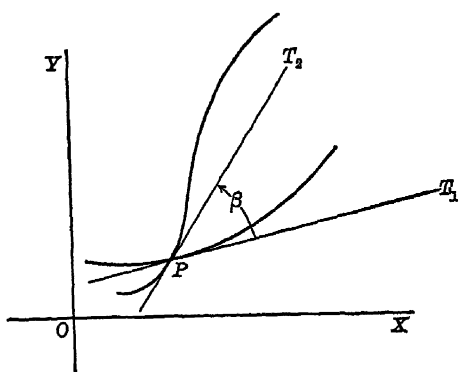

FIG. 53a.

Let m_1 and m_2 be the slopes of two curves at a point of intersection. It is shown in analytic geometry that the angle β from a line with slope m_1 to one with slope m_2 satisfies the equation

$$\tan \beta = \frac{m_2 - m_1}{1 + m_1m_2}. \tag{50}$$

This equation thus gives the angle β from a curve with slope m_1 to one with slope m_2, the angle being considered positive when measured in the counter-clockwise direction.

Example. Find the angles determined by the line $y = x$ and the parabola $y = x^2$.

Solving the equations simultaneously, we find that the line and parabola intersect at (1, 1) and (0, 0). The slope of the line is 1. The slope at any point of the parabola is

$$\frac{dy}{dx} = 2x.$$

At (1, 1) the slope of the parabola is then 2 and the angle from the line to the parabola is then given by

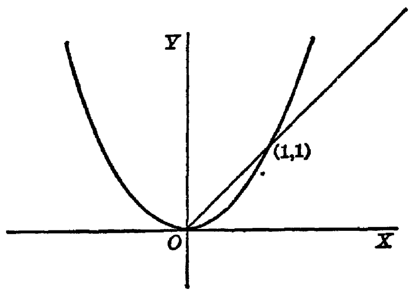

FIG. 53b.

$$\tan \beta_1 = \frac{2 - 1}{1 + 2} = \frac{1}{3},$$

whence

$$\beta_1 = \tan^{-1}\tfrac{1}{3} = 18^\circ\, 26'.$$

At (0, 0) the slope of the parabola is 0 and so the angle from the line to the parabola is given by the equation

$$\tan \beta_2 = \frac{0 - 1}{1 + 0} = -1,$$

whence

$$\beta_2 = -45^\circ.$$

The negative sign signifies that the angle is measured in the clockwise direction from the line to the parabola.

EXERCISES

Find the tangent and normal to each of the following curves at the point indicated:

1. The circle $x^2 + y^2 = 5$ at $(1, -2)$.
2. The hyperbola $xy = 4$ at $(2, 2)$.
3. The parabola $y^2 = ax$ at $(a, -a)$.
4. The exponential curve $y = ab^x$ at $x = 0$.
5. The sine curve $y = \frac{1}{2}\sin 2x$ at $x = \frac{\pi}{2}$.
6. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

7. The hyperbola $x^2 - xy - y^2 = 2x$ at (2, 0).
8. The semicubical parabola $y^3 = x^2$ at $(-1, 1)$.
9. Find the equation of the normal to the cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi)$$

at the point where $\phi = \phi_1$. Show that it passes through the point where the rolling circle touches the x-axis.

Find the angles at which the following pairs of curves intersect:

10. $y^2 = 2x$, $x^2 = 2y$.
11. $x^2 + y^2 = 4$, $x^2 + y^2 - 4x = 4$.
12. $x^2 + y^2 - 2x = 3$, $y^2 = 4x$.
13. $y = \sin x$, $y = \cos x$.
14. $y = \ln x$, $y = x - 1$.
15. $y = \sin 2x$, $y = \sin x$.
16. Show that for all values of the constants a and b the curves

$$x^2 - y^2 = a^2, \qquad xy = b^2$$

intersect at right angles.

17. Show that the curves

$$y = e^{ax}, \qquad y = e^{ax} \sin bx$$

are tangent at each point of intersection.

18. Show that the part of the tangent to the hyperbola $xy = a^2$ included between the coordinate axes is bisected at the point of tangency.

19. Let the normal to the parabola $y^2 = ax$ at P cut the x-axis at N. Show that the projection of PN on the x-axis has a constant length.

20. The focus of the parabola $y^2 = 4ax$ is the point $F(a, 0)$. Show that the tangent at any point P on the parabola makes equal angles with FP and the line through P parallel to the x-axis.

21. Let P be any point on the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right),$$

M the projection of P on the x-axis, and N the projection of M on the tangent at P. Show that MN is constant in length.

22. Show that the angle between the tangent at any point P and the line joining P to the origin is the same at all points of the curve

$$\ln(x^2 + y^2) = k \tan^{-1}\frac{y}{x}.$$

54. Direction of Curvature. — A curve is said to be *concave upward* at a point P if the part of the curve near P lies

above the tangent at P. It is *concave downward* at Q if the part near Q lies below the tangent at Q.

At points where $\frac{d^2y}{dx^2}$ *is positive, the curve is concave upward; where* $\frac{d^2y}{dx^2}$ *is negative, the curve is concave downward.*

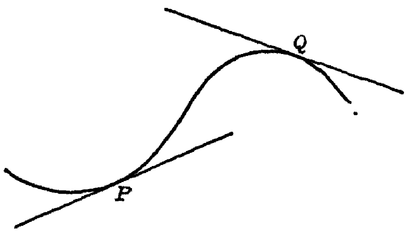

FIG. 54

For

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

If then $\frac{d^2y}{dx^2}$ is positive, by Art. 11, $\frac{dy}{dx}$, the slope, increases as x increases and decreases as x decreases. The curve therefore rises above the tangent on both sides of the point. If, however, $\frac{d^2y}{dx^2}$ is negative, the slope decreases as x increases and increases as x decreases, and so the curve falls below the tangent.

55. Point of Inflection. — A point like A (Fig. 55a), on one side of which the curve is concave upward, on the other

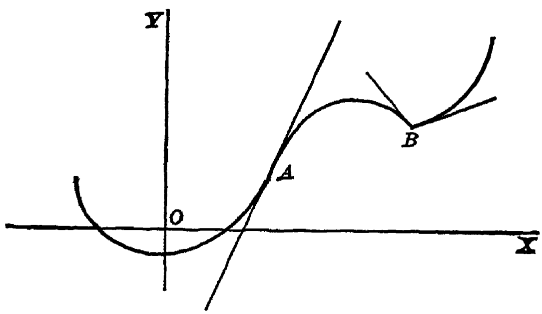

FIG. 55a.

concave downward, is called a *point of inflection.* It is assumed that there is a definite tangent at the point of inflection. A point like B is not called a point of inflection.

The second derivative is positive on one side of a point of

inflection and negative on the other. Ordinary functions change sign only by passing through zero or infinity. Hence to find points of inflection we find where $\frac{d^2y}{dx^2}$ is zero or infinite. If the second derivative changes sign at such a point, it is a point of inflection. If the second derivative has the same sign on both sides, it is not a point of inflection.

Example 1. Examine the curve $3y = x^4 - 6x^2$ for direction of curvature and points of inflection.

The second derivative is

$$\frac{d^2y}{dx^2} = 4(x^2 - 1).$$

This is zero at $x = \pm 1$. It is positive and the curve concave upward on the left of $x = -1$ and on the right of $x = +1$. It is negative and the curve concave downward between $x = -1$ and $x = +1$. The second derivative changes sign at $A\ (-1, -\frac{5}{3})$ and $B\ (+1, -\frac{5}{3})$, which are therefore points of inflection (Fig. 55b).

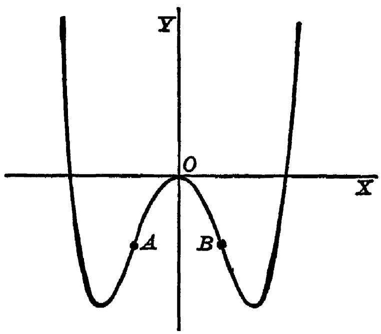

FIG. 55b.

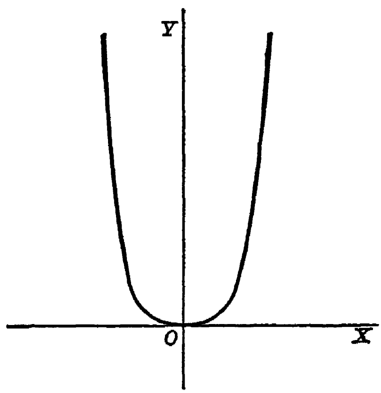

FIG. 55c.

Example 2. Examine the curve $y = x^4$ for points of inflection. In this case the second derivative is

$$\frac{d^2y}{dx^2} = 12x^2.$$

This is zero when x is zero but is positive for all other values of x. The second derivative does not change sign and there is consequently no point of inflection (Fig. 55c).

Example 3. If $x > 0$, show that $\sin x > x - \frac{x^3}{3!}$.

Let

$$y = \sin x - x + \frac{x^3}{3!}.$$

We are to show that $y > 0$. Differentiation gives

$$\frac{dy}{dx} = \cos x - 1 + \frac{x^2}{2!}, \qquad \frac{d^2y}{dx^2} = -\sin x + x.$$

When x is positive, $\sin x$ is less than x and so $\frac{d^2y}{dx^2}$ is positive. Therefore $\frac{dy}{dx}$ increases with x. Since $\frac{dy}{dx}$ is zero when x is zero, $\frac{dy}{dx}$ is then positive when $x > 0$, and so y increases with x. Since $y = 0$ when $x = 0$, y is therefore positive when $x > 0$, which was to be proved.

EXERCISES

Examine the following curves for direction of curvature and points of inflection:

1. $y = x^3 - 3x + 2$.
2. $y = 2x^3 - 6x^2$.
3. $y = x^4 + 4x^3 + 6x^2$.
4. $y^3 = x - 1$.
5. $y = xe^{-x}$.
6. $y = e^{-x^2}$.
7. $x^2y - 4x + 3y = 0$.
8. $y^3 = 3x^2$.

Prove the following inequalities:

9. $\cos x > 1 - \frac{x^2}{2}$.

10. $e^x > 1 + x + \frac{x^2}{2}$, if $x > 0$.

11. $x \ln x - \frac{x^2}{2} + \frac{1}{2} < 0$, if $0 < x < 1$.

12. $\ln \sec x > \frac{x^2}{2}$, if $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

13. According to Van der Waal's equation, the pressure p and volume v of a gas at constant temperature T are connected by the equation

$$p = \frac{RT}{m(v - b)} - \frac{a}{v^2},$$

a, b, m, and R being constants. If p is taken as ordinate and v as abscissa, the curve represented by this equation has a point of inflection. The value of T for which the tangent at the point of inflection is horizontal is called the critical temperature. Show that the critical temperature is

$$T = \frac{8\ am}{27\ Rb}.$$

56. Length of a Plane Curve. — The length of an arc PQ of a curve is defined as the limit (if there is a limit) approached by the length of a broken line with vertices on PQ as the number of its sides increases indefinitely, their lengths approaching zero.

We shall now show that if the slope of a curve is continuous the ratio of a chord to the arc it subtends approaches 1 as the chord approaches zero.

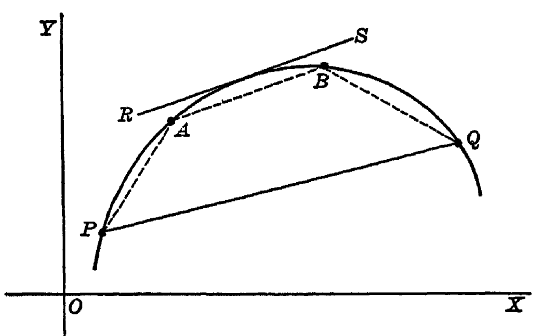

FIG. 56.

In the arc PQ (Fig. 56) inscribe a broken line $PABQ$. Projecting on PQ, we get

$$PQ = \text{proj.}\ PA + \text{proj.}\ AB + \text{proj.}\ BQ.$$

The projection of a chord, such as AB, is equal to the product of its length by the cosine of the angle it makes with PQ.

On the arc AB is a tangent RS parallel to AB. Let α be the largest angle that any tangent on the arc PQ makes with the chord PQ. The angle between RS and PQ is not greater than α. Consequently, the angle between AB and PQ is not greater than α. Therefore

$$\text{proj. } AB \geqq AB \cos \alpha.$$

Similarly,

$$\text{proj. } PA \geqq PA \cos \alpha,$$
$$\text{proj. } BQ \geqq BQ \cos \alpha.$$

Adding these equations, we get

$$PQ \geqq (PA + AB + BQ) \cos \alpha.$$

It is evident that this result can be extended to a broken line with any number of sides. As the number of sides increases indefinitely, the expression in parenthesis approaches the length of the arc PQ. Therefore

$$PQ \geqq \text{arc } PQ \cos \alpha,$$

that is,

$$\frac{\text{chord } PQ}{\text{arc } PQ} \geqq \cos \alpha.$$

If the slope of the curve is continuous, the angle α approaches zero as Q approaches P. Hence $\cos \alpha$ approaches 1 and

$$\lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} \geqq 1.$$

Since the chord is always less than the arc, the limit cannot be greater than 1. Therefore, finally,

$$\lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} = 1. \tag{56}$$

57. Differential of Arc. — Let s be the distance measured along a curve from a fixed point A to a variable point P.

Then s is a function of the coördinates of P. Let ϕ be the angle from the positive direction of the x-axis to the tangent PT drawn in the direction of increasing s.

If P moves to a neighboring position Q, the increments in x, y, and s are

$$\Delta x = PR, \qquad \Delta y = RQ, \qquad \Delta s = \text{arc } PQ.$$

From the figure it is seen that

$$\cos(RPQ) = \frac{\Delta x}{PQ} = \frac{\Delta x}{\Delta s}\frac{\Delta s}{PQ},$$
$$\sin(RPQ) = \frac{\Delta y}{PQ} = \frac{\Delta y}{\Delta s}\frac{\Delta s}{PQ}.$$

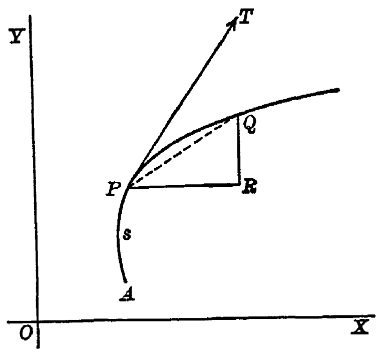

FIG. 57a.

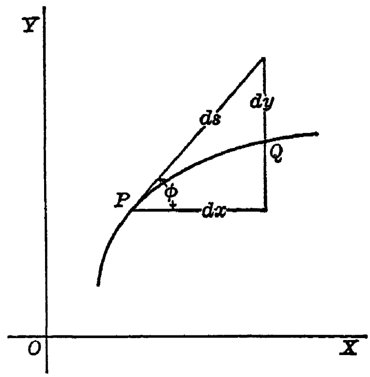

FIG. 57b.

As Q approaches P, RPQ approaches ϕ and

$$\frac{\Delta s}{PQ} = \frac{\text{arc } PQ}{\text{chord } PQ}$$

approaches 1. The above equations then give in the limit

$$\cos\phi = \frac{dx}{ds}, \qquad \sin\phi = \frac{dy}{ds}. \tag{57a}$$

These equations express that dx and dy are the sides of a right triangle with hypotenuse ds extending along the tangent

(Fig. 57b). All the equations connecting dx, dy, ds, and ϕ can be read off this triangle. One of particular importance is

$$ds^2 = dx^2 + dy^2. \tag{57b}$$

58. Curvature. — If an arc is everywhere concave toward its chord, the amount it is bent can be measured by the angle β between the tangents at its ends. The ratio

$$\frac{\beta}{\text{arc } PP'} = \frac{\phi' - \phi}{\Delta s} = \frac{\Delta\phi}{\Delta s}$$

is the average bending per unit length along PP' (Fig. 58a). The limit as P' approaches P,

$$\lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds},$$

is called the *curvature* at P. It is greater where the curve bends more sharply, less where it is more nearly straight.

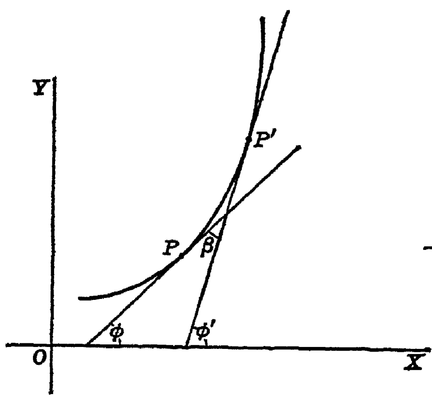

FIG. 58a.

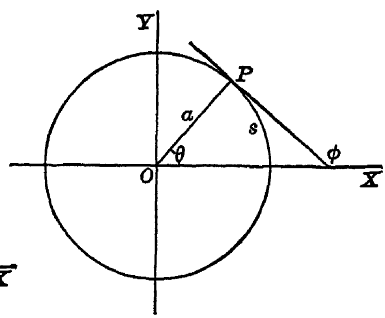

FIG. 58b.

In case of a circle (Fig. 58b)

$$\phi = \theta + \frac{\pi}{2}, \qquad s = a\theta.$$

Consequently,

$$\frac{d\phi}{ds} = \frac{d\theta}{a\,d\theta} = \frac{1}{a},$$

that is, *the curvature of a circle is constant and equal to the reciprocal of its radius.*

59. Radius of Curvature. — We have just seen that the radius of a circle is the reciprocal of its curvature. The *radius of curvature* of any curve is defined as the reciprocal of its curvature, that is,

$$\text{radius of curvature} = \rho = \frac{ds}{d\phi}. \tag{59a}$$

It is the radius of the circle which has the same curvature as the given curve at the given point.

To express ρ in terms of x and y we note that

$$\phi = \tan^{-1}\frac{dy}{dx}.$$

Consequently

$$d\phi = \frac{1}{1+\left(\frac{dy}{dx}\right)^2} d\left(\frac{dy}{dx}\right) = \frac{\frac{d^2y}{dx^2}dx}{1+\left(\frac{dy}{dx}\right)^2}.$$

Also
$$ds = \sqrt{dx^2 + dy^2}.$$

Substituting these values for ds and $d\phi$, we get

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \tag{59b}$$

If the radical in the numerator is taken positive, ρ will have the same sign as $\frac{d^2y}{dx^2}$, that is, the radius will be positive when the curve is concave upward. If merely the numerical value is wanted, the sign can be omitted.

By a similar proof we could show that

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}} \tag{59c}$$

Example 1. Find the radius of curvature of the parabola $y^2 = 4x$ at the point (4, 4).

At the point (4, 4) the derivatives have the values

$$\frac{dy}{dx} = \frac{2}{y} = \frac{1}{2}, \qquad \frac{d^2y}{dx^2} = -\frac{4}{y^3} = -\frac{1}{16}.$$

Therefore

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(1 + \frac{1}{4}\right)^{\frac{3}{2}}}{-\frac{1}{16}} = -10\sqrt{5}.$$

The negative sign shows that the curve is concave downward. The length of the radius is $10\sqrt{5}.$

Example 2. Find the radius of curvature of the curve represented by the polar equation $r = a\cos\theta$.

The expressions for x and y in terms of θ are

$$x = r\cos\theta = a\cos\theta\cos\theta = a\cos^2\theta,$$
$$y = r\sin\theta = a\cos\theta\sin\theta.$$

Consequently,

$$\frac{dy}{dx} = \frac{a(\cos^2\theta - \sin^2\theta)}{-2a\cos\theta\sin\theta} = \frac{a\cos 2\theta}{-a\sin 2\theta} = -\cot 2\theta,$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = -\frac{2\csc^2 2\theta\, d\theta}{a\sin 2\theta\, d\theta} = -\frac{2}{a}\csc^3 2\theta.$$

$$\rho = \frac{[1 + \cot^2 2\theta]^{\frac{3}{2}}}{-\frac{2}{a}\csc^3 2\theta} = -a\frac{(\csc^2 2\theta)^{\frac{3}{2}}}{2\csc^3 2\theta} = -\frac{a}{2}.$$

The radius is thus constant. The curve is in fact a circle.

60. Center and Circle of Curvature. — At each point of a curve is a circle on the concave side tangent at the point with radius equal to the radius of curvature. This circle is

called the *circle of curvature.* Its center is called the *center of curvature.*

Since the circle and curve are tangent at P, they have the same slope $\dfrac{dy}{dx}$ at P. Since they have the same radius of

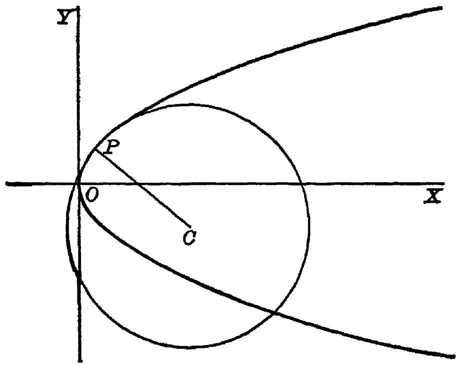

FIG. 60

curvature, the second derivatives will also be equal at P. The circle of curvature is thus the circle through P such that $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ have the same values for the circle as for the curve at P.

EXERCISES

1. The length of arc measured from a fixed point on a certain curve is $s = x^2$. Find the slope of the curve at $x = 2$.

2. Can $x = \cos s$ represent a curve on which s is the length of arc measured from a fixed point? Can $x = \tan s$ represent such a curve?

3. If $s = kx$, where k is constant, show that the curve is a straight line.

Find the radius of curvature on each of the following curves at the point indicated:

4. $xy = 4$, at $(2, 2)$.

5. $x^2 + xy + y^2 = 3$, at $(1, 1)$.

6. $x^2 - y^2 = 1$, at $(1, 0)$.

7. $r = a(1 + \cos\theta)$, at $\theta = \dfrac{\pi}{2}$.

Find an expression for the radius of curvature at any point of each of the following curves:

8. $x = \frac{1}{4} y^2 - \frac{1}{2} \ln y$.
9. $y = \ln \sec x$.
10. $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.
11. $x = a(\phi - \sin \phi)$, $y = a(1 - \cos \phi)$.
12. Show that the radius of curvature at a point of inflection is infinite.
13. A string held taut is unwound from a fixed circle. The end of the string generates a curve with parametric equations

$$x = a \cos \theta + a \theta \sin \theta, \quad y = a \sin \theta - a \theta \cos \theta,$$

a being the radius of the circle and θ the angle subtended at the center by the part unwound. Show that the center of curvature corresponding to any point on the path is the point where the string is tangent to the circle.

14. Show that the radius of curvature at any point (x, y) of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is three times the perpendicular from the origin to the tangent at (x, y).

61. Sine and Cosine of Small Angle. — In Art. 33, we proved that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{61a}$$

Combining this with the equations

$$\frac{\sin x - x}{x} = \frac{\sin x}{x} - 1,$$

$$\frac{1 - \cos x}{x} = \frac{2 \sin^2 \frac{x}{2}}{x} = \sin \frac{x}{2} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right),$$

we find

$$\lim_{x \to 0} \frac{\sin x - x}{x} = 0, \tag{61b}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \tag{61c}$$

Equations (61b) and (61c) show that if x is infinitesimal $\sin x - x$ and $1 - \cos x$ are infinitesimals of higher order than x. In a fraction with denominator x we can then replace $\sin x$ by x and $\cos x$ by 1 without changing the value of the limit.

62. Derivatives of Arc in Polar Coördinates. — The angle from the outward drawn radius to the tangent drawn in the direction of increasing s is usually represented by the letter ψ.

Let r, θ be the polar coördinates of P, and $r + \Delta r$, $\theta + \Delta\theta$ those of Q (Fig. 62a). Draw QR perpendicular to PR and let $\Delta s = \text{arc } PQ$. Then

$$\sin(RPQ) = \frac{RQ}{PQ} = \frac{(r+\Delta r)\sin\Delta\theta}{PQ} = (r+\Delta r)\frac{\sin\Delta\theta}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta s}\cdot\frac{\Delta s}{PQ}.$$

$$\begin{aligned}\cos(RPQ) &= \frac{PR}{PQ} = \frac{(r+\Delta r)\cos\Delta\theta - r}{PQ}\\ &= \cos(\Delta\theta)\frac{\Delta r}{PQ} - \frac{r(1-\cos\Delta\theta)}{PQ}\\ &= \cos(\Delta\theta)\frac{\Delta r}{\Delta s}\cdot\frac{\Delta s}{PQ} - \frac{r(1-\cos\Delta\theta)}{\Delta\theta}\,\frac{\Delta\theta}{\Delta s}\cdot\frac{\Delta s}{PQ}.\end{aligned}$$

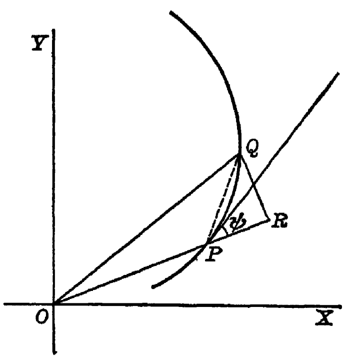

FIG. 62a.

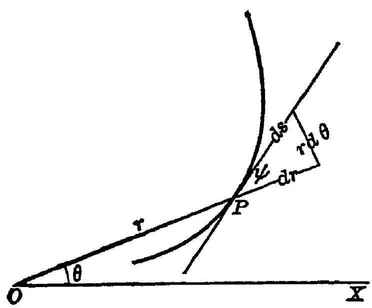

FIG. 62b.

As $\Delta\theta$ approaches zero,

$$\lim(RPQ) = \psi,\ \lim\frac{\sin\Delta\theta}{\Delta\theta} = 1,\ \lim\frac{1-\cos\Delta\theta}{\Delta\theta} = 0,\ \lim\frac{\Delta s}{PQ} = 1.$$

The above equations then give in the limit,

$$\sin\psi = \frac{r\,d\theta}{ds}, \qquad \cos\psi = \frac{dr}{ds}. \tag{62a}$$

These limits could be obtained a little more quickly by replacing $\sin\Delta\theta$ by $\Delta\theta$ and $\cos\Delta\theta$ by 1 in the expressions for $\sin(RPQ)$ and $\cos(RPQ)$ as suggested in Art. 61.

Equations (62a) show that dr and $r\,d\theta$ are the sides of a right triangle with hypotenuse ds and base angle ψ. From this triangle all the equations connecting dr, $d\theta$, ds, and ψ can be obtained. The most important of these are

$$\tan\psi = \frac{r\,d\theta}{dr}, \qquad ds^2 = dr^2 + r^2d\theta^2. \tag{62b}$$

Example. The logarithmic spiral $r = ae^\theta$.
In this case, $dr = ae^\theta\,d\theta$ and so

$$\tan\psi = \frac{r\,d\theta}{dr} = 1.$$

The angle ψ is therefore constant and equal to 45 degrees. The equation

$$\cos\psi = \frac{dr}{ds} = \frac{1}{\sqrt{2}}$$

shows that $\frac{dr}{ds}$ is also constant and so r and s increase proportionally.

EXERCISES

Find the angle ψ at the point indicated on each of the following curves:

1. The spiral $r = a\theta$, at $\theta = 1$.
2. The circle $r = a\sin\theta$, at $\theta = \frac{\pi}{4}$.
3. The straight line $r = a\sec\theta$, at $\theta = \frac{\pi}{6}$.

4. The parabola $r(1 - \cos\theta) = k$, at $\theta = \frac{\pi}{2}$.

5. The lemniscate $r^2 = 2\,a^2 \cos 2\,\theta$, at $\theta = \frac{5}{8}\pi$.

6. Show that the curves $r = ae^{\theta}$ and $r = ae^{-\theta}$ are perpendicular at each of their points of intersection.

7. Find the angles at which the curves $r = a\cos\theta$, $r = \frac{1}{2}\,a$ intersect.

8. Find the points on the cardioid $r = a(1 - \cos\theta)$ where the tangent is parallel to the initial line.

9. Let $P(r, \theta)$ be a point on the hyperbola $r^2 \sin 2\,\theta = c$. Show that the triangle formed by the radius OP, the tangent at P, and the initial line is isosceles.

10. Find the slope of the curve $r = a \sin 2\,\theta$ at the point $\theta = \frac{\pi}{6}$.

CHAPTER IX

VELOCITY AND ACCELERATION IN A CURVED PATH

63. Speed of a Particle. — When a particle moves along a curve, its speed is the rate of change of distance along the path.

Let a particle P move along the curve AB (Fig. 63). Let s be the arc from a fixed point A to P. The speed of the particle is then

$$v = \frac{ds}{dt}. \tag{63}$$

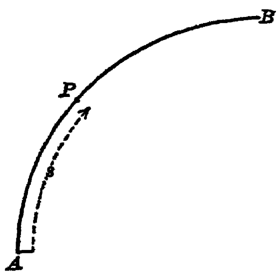

FIG. 63.

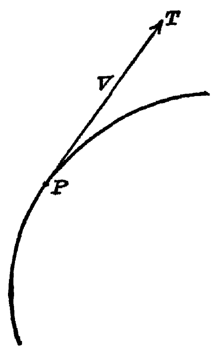

FIG. 64.

64. Velocity of a Particle. — The velocity of a particle at the point P in its path is defined as the vector* PT tangent to the path at P, drawn in the direction of motion with length equal to the speed at P. To specify the velocity we must then give the speed and direction of motion.

* A vector is a quantity having length and direction. The direction is usually indicated by an arrow. Two vectors are called equal when they extend along the same line or along parallel lines and have the same length and direction.

The particle can be considered as moving instantaneously in the direction of the tangent. The velocity indicates in magnitude and direction the displacement it would experience in a unit of time if its speed and direction of motion did not change.

65. Components of Velocity. — The projection of a vector on a line is called its *component* along that line. If a positive direction is assigned along the line, the component is considered positive when it extends in that direction and negative when it extends in the opposite direction.

To specify a velocity in a plane it is customary to give its components along the coördinate axes. If PT is the velocity at P (Fig. 65a), the x-component is

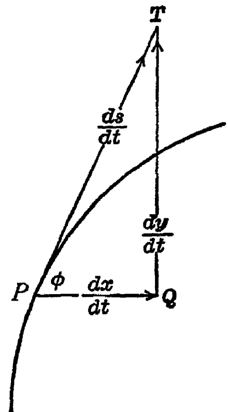

FIG. 65a.

$$PQ = PT \cos\phi = \frac{ds}{dt}\cos\phi = \frac{ds}{dt}\frac{dx}{ds} = \frac{dx}{dt},$$

and the y-component is

$$QT = PT \sin\phi = \frac{ds}{dt}\sin\phi = \frac{ds}{dt}\frac{dy}{ds} = \frac{dy}{dt}.$$

The components are thus the rates of change of the coordinates.

Since

$$PT^2 = PQ^2 + QT^2,$$

the speed is expressed in terms of the components by the equation

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2.$$

Example 1. Neglecting the resistance of the air, a bullet fired with a velocity of 1000 ft. per second at an angle of 30° with the horizontal will move a horizontal distance

$$x = 500\,t\sqrt{3}$$

and a vertical distance

$$y = 500\,t - 16.1\,t^2$$

in t seconds. Find its velocity and speed at the end of 10 seconds.

At time t the components of velocity are

$$\frac{dx}{dt} = 500\sqrt{3}, \qquad \frac{dy}{dt} = 500 - 32.2\,t.$$

When $t = 10$ these have the values

$$\frac{dx}{dt} = 500\sqrt{3}, \qquad \frac{dy}{dt} = 178$$

and the speed is then

$$\frac{ds}{dt} = \sqrt{(500\sqrt{3})^2 + (178)^2} = 884 \text{ ft./sec.}$$

Example 2. The radius OP of a circle rotates about the center O with constant angular velocity

$$\frac{d\theta}{dt} = \omega.$$

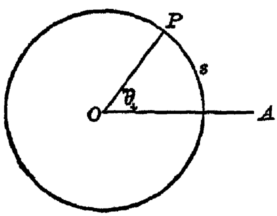

FIG. 65b.

Find the velocity of P.

Using rectangular coördinates with the origin at the center of the circle (Fig. 65b), the coördinates of P are

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The components of velocity are

$$\frac{dx}{dt} = -r\sin\theta\,\frac{d\theta}{dt} = -r\omega\sin\theta,$$

$$\frac{dy}{dt} = r\cos\theta\,\frac{d\theta}{dt} = r\omega\cos\theta.$$

The speed of the point P is

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = r\omega.$$

66. Composition of Velocities. — Acceleration is defined as the rate of change of velocity. To use this definition it is necessary to know what is meant by the difference of two velocities. Since velocities have directions as well as magnitudes, the mere subtraction of magnitudes gives no result of particular value.

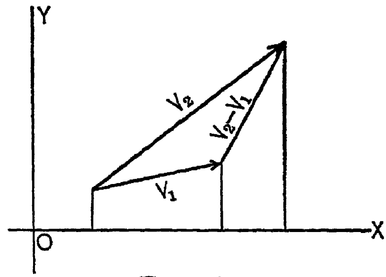

FIG. 66a

Let V_1 and V_2 be two vectors. If they do not extend from a common point, construct two new vectors having the same magnitudes and directions and extending from a common point (Fig. 66a). Then $V_2 - V_1$ is defined as the vector extending from the end of V_1 to the end of V_2.

It is clear from the diagram that the projections on the x-axis satisfy the relation

$$\text{proj. } (V_2 - V_1) = \text{proj. } V_2 - \text{proj. } V_1$$

and a similar relation holds for any other axis. That is, *each component of $V_2 - V_1$ is equal to the difference of the corresponding components of V_2 and V_1.*

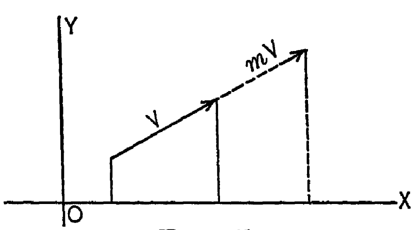

FIG. 66b.

If m is a number and V a vector, the notation mV is used to represent a vector m times as long as V and extending in the same direction if m is positive, the opposite direction if m is negative. It is evident from Fig. 66b that the components of mV are m times those of V.

The quotient $\dfrac{V}{m}$ can be considered as a product $\dfrac{1}{m}V$. Its components are obtained by dividing those of V by m.

67. Acceleration. — Let a particle moving along a curve reach P (x, y) at time t and P' $(x + \Delta x, y + \Delta y)$ at time $t + \Delta t$. Let V be the velocity at P and

$$V' = V + \Delta V$$

the velocity at P'. The acceleration of the particle when passing through P is defined as the limit

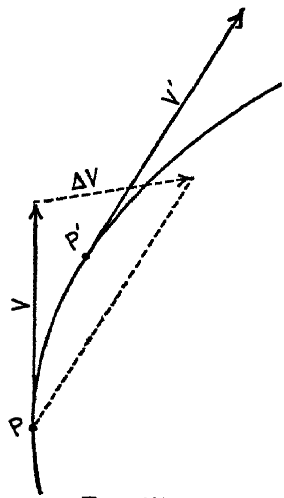

FIG. 67a.

$$A = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt}.$$

In this equation ΔV is the vector equal to the difference of the velocities at P and P' and

$$\frac{\Delta V}{\Delta t}$$

has components equal to those of ΔV divided by Δt.

To find the components of acceleration let

$$\frac{dx}{dt} = V_x, \qquad \frac{dy}{dt} = V_y.$$

Then the components of V are

$$V_x, \qquad V_y,$$

and those of V' are

$$V_x + \Delta V_x, \qquad V_y + \Delta V_y.$$

Hence the components of $\Delta V = V' - V$ are

$$\Delta V_x, \qquad \Delta V_y,$$

and those of $\dfrac{\Delta V}{\Delta t}$ are

$$\frac{\Delta V_x}{\Delta t}, \qquad \frac{\Delta V_y}{\Delta t}.$$

As Δt approaches zero these approach the components A_x, A_y of acceleration. Hence

$$\left.\begin{aligned} A_x &= \frac{dV_x}{dt} = \frac{d^2x}{dt^2} \\ A_y &= \frac{dV_y}{dt} = \frac{d^2y}{dt^2} \end{aligned}\right\}. \tag{67}$$

The components of acceleration with respect to rectangular coördinates are thus the second derivatives of the coördinates with respect to the time.

Example 1. A particle describes an ellipse in such a way that its coördinates at time t are

$$x = a \cos kt, \qquad y = b \sin kt,$$

a, b, k being constants. Find its acceleration.

The components of acceleration are

$$\frac{d^2x}{dt^2} = -ak^2 \cos kt = -k^2x = -k^2r \cos\theta,$$
$$\frac{d^2y}{dt^2} = -ak^2 \sin kt = -k^2y = -k^2r \sin\theta,$$

where r, θ are the polar coördinates of P (Fig. 67b). These are the components of a vector of length k^2r directed toward the origin along the line OP. The acceleration is therefore directed toward the origin and at each instant is proportional to the distance r of the moving particle from the origin.

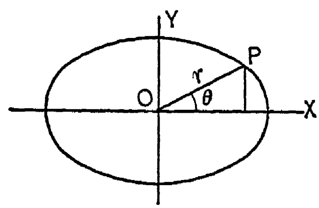

FIG. 67b.

Example 2. A particle describes the ellipse

$$x^2 + 2y^2 = 4$$

with constant speed v. Find its velocity and acceleration at the instant when it passes through the vertex $x = 2$, $y = 0$.

Differentiating with respect to t, we get

$$2x\frac{dx}{dt} + 4y\frac{dy}{dt} = 0,$$

whence

$$\frac{dx}{dt} = -\frac{2y}{x}\frac{dy}{dt}.$$

Substituting this value, we find

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\frac{x^2 + 4y^2}{x^2}}\,\frac{dy}{dt}$$

and so

$$\frac{dy}{dt} = \frac{vx}{\sqrt{x^2 + 4y^2}}, \quad \frac{dx}{dt} = -\frac{2vy}{\sqrt{x^2 + 4y^2}}.$$

Differentiating again with respect to t,

$$\frac{d^2y}{dt^2} = \frac{4vy\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)}{(x^2 + 4y^2)^{\frac{3}{2}}} = -\frac{16v^2y}{(x^2 + 4y^2)^2},$$

$$\frac{d^2x}{dt^2} = \frac{2vx\left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)}{(x^2 + 4y^2)^{\frac{3}{2}}} = -\frac{8v^2x}{(x^2 + 4y^2)^2}.$$

When $x = 2$, $y = 0$, the components of velocity are

$$\frac{dx}{dt} = 0, \qquad \frac{dy}{dt} = v,$$

and the components of acceleration

$$\frac{d^2x}{dt^2} = -v^2, \qquad \frac{d^2y}{dt^2} = 0.$$

EXERCISES

1. At time t the coördinates of a moving point are

$$x = 3t + 10, \qquad y = 2t^2 - 4t.$$

Find its velocity and speed at the time $t = 2$.

2. A particle describes the curve

$$x = e^t \cos t, \qquad y = e^t \sin t.$$

Find its velocity and speed at time t.

3. Two particles (x_1, y_1), (x_2, y_2) move in such a way that

$$x_1 = 1 + 2t, \qquad y_1 = 2 - 3t^2,$$
$$x_2 = 3 + 2t^2, \qquad y_2 = 5 - 4t^3.$$

Find the two velocities and show that they are always parallel.

4. A rod of length a slides with its ends in the coördinate axes. If the end in the x-axis moves with constant speed v, find the velocity and speed of its middle point at the instant when the rod makes an angle of 30° with the x-axis.

5. A particle moves along the parabola $y^2 = 4ax$ with constant speed v. Find the components of its velocity at the instant when $x = a$, $y = 2a$.

6. At time t the coordinates of a moving particle are

$$x = a_1t^2 + b_1t + c_1,$$
$$y = a_2t^2 + b_2t + c_2,$$

$a_1, b_1, c_1, a_2, b_2, c_2$ being constants. Show that its acceleration is constant.

7. A particle moves with constant speed around a circle of radius a. Find its acceleration.

8. A particle describes the hyperbola

$$x = \tfrac{1}{2}(e^{kt} + e^{-kt}), \qquad y = \tfrac{1}{2}(e^{kt} - e^{-kt}).$$

Show that its acceleration is always directed away from the origin and is in magnitude proportional to the distance from the origin.

9. A particle moves along the parabola $x^2 = 2ay$ in such a way that $\frac{dx}{dt}$ is constant. Show that its acceleration is constant.

10. A particle describes the parabola $y^2 = 4ax$ with constant speed v. Find its acceleration at the vertex.

11. When a wheel rolls along a straight line a point P on its perimeter describes the cycloid with parametric equations

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi),$$

a being the radius and ϕ the angle through which it has turned. If the wheel rotates with constant angular velocity

$$\frac{d\phi}{dt} = \omega,$$

find the velocity and acceleration of the point at the instant when it passes through its lowest position.

12. If a string is held taut while it is unwound from a fixed circle, its end describes the curve

$$x = a\cos\theta + a\theta\sin\theta, \qquad y = a\sin\theta - a\theta\cos\theta,$$

a being the radius of the circle and θ the angle at the center subtended by the arc unwound. Show that the end moves at each instant with the velocity it would have if the straight part of the string $a\theta$ rotated with angular velocity $\omega = \dfrac{d\theta}{dt}$ about the point where it meets the fixed circle (the instantaneous center of rotation). Does the acceleration have a similar property?

13. A particle $P(r, \theta)$ describes the circle $r = 2a\cos\theta$, the radius vector OP rotating with constant angular velocity ω. Find its velocity and acceleration and by projecting on OP determine the components of velocity and acceleration along that line. Are these equal to $\dfrac{dr}{dt}$ and $\dfrac{d^2r}{dt^2}$?

14. A piece of mechanism consists of a rod rotating with constant angular velocity ω about one end and a ring sliding along the rod with constant speed v. (1) If when $t = 0$ the ring is at the center of rotation, find its position, velocity, and acceleration at time $t = t_1$. (2) Find the velocity and acceleration immediately after $t = t_1$ if at that instant the rod ceases to rotate but the ring continues to slide with unchanged speed along the rod. (3) Find the velocity and acceleration immediately after $t = t_1$ if at that instant the ring ceases sliding but the rod continues rotating. (4) How are these three velocities related? How are the three accelerations related?

CHAPTER X

INTEGRATION

68. Integral. — In many problems the derivative or differential of a quantity is known and it is necessary to find the function of which it is the derivative or differential. Thus, if the speed of a moving particle at time t is a known function

$$v = \frac{ds}{dt} = f(t),$$

the distance traversed in time t is the quantity s which has $f(t)$ as derivative. To find this distance we must then determine a function with derivative equal to $f(t)$.

The process of finding a function with a given differential is called *integration*. If

$$dF(x) = f(x)\,dx,$$

then $F(x)$ is called an integral of $f(x)\,dx$ and this is expressed by the notation

$$F(x) = \int f(x)\,dx.$$

By differentiation we pass from the function to its differential. By integration we pass from the differential to the function. Integration is thus the inverse of differentiation.

For example, since $d(x^2) = 2\,x\,dx$,

$$\int 2\,x\,dx = x^2.$$

Similarly

$$\int \cos x\,dx = \sin x, \qquad \int e^x\,dx = e^x.$$

The test of integration is to differentiate the answer. If the integration is correct, the differential of the answer must equal the expression to be integrated.

It should be noted that we always integrate a differential and not a derivative. There are several reasons for this. A very important one is the fact that the differential of a quantity is independent of the variable in terms of which it is expressed. If the same quantity y can be expressed as a function of two different variables,

$$y = f_1(x), \qquad y = f_2(t),$$

its differential has equal values whether the one or the other of these is used. But the derivatives of y with respect to x and t are not in general equal. If integration were defined as the process of finding a function with a given derivative, there would be a different integral for every variable.

69. Constant of Integration. — In solving a problem having an integral as answer it is necessary to inquire how many integrals a given differential has. For, if there are several different integrals, we cannot be sure the one we find is the answer.

As a matter of fact each differential has an infinite number of integrals but they are related in a rather simple way. If

$$dF(x) = f(x)\,dx,$$

and C is any constant,

$$d[F(x) + C] = dF(x) = f(x)\,dx.$$

If then $F(x)$ is one integral of a given differential,

$$F(x) + C$$

is another.

Conversely, *if two continuous functions are integrals of the same differential, their difference is constant.*

To prove this, suppose $F_1(x)$ and $F_2(x)$ are both integrals of $f(x)\,dx$. Then

$$dF_1(x) = dF_2(x) = f(x)\,dx.$$

Let

$$y = F_2(x) - F_1(x),$$

and plot the locus representing y as a function of x. The slope of this locus is

$$\frac{dy}{dx} = \frac{dF_2(x)}{dx} - \frac{dF_1(x)}{dx} = 0.$$

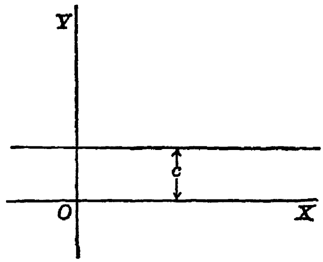

FIG. 69

Since the slope is everywhere zero, the locus is a horizontal line. The equation of such a line is $y = C$. Therefore

$$y = F_2(x) - F_1(x) = C,$$

which was to be proved.

If then $F(x)$ is one continuous integral of a given diffeiential any other has the form

$$\int f(x)\,dx = F(x) + C.$$

Any constant value can be assigned to C. It is called an *arbitrary constant.* In the statement of a definite problem there is usually some information by which this constant can be determined.

70. Integration Formulas. — Let a, n be constants and u, v, w functions of a single independent variable.

I. $\displaystyle\int du + dv + dw = \int du + \int dv + \int dw.$

II. $\displaystyle\int a\,du = a\int du.$

III. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C$, if n is not -1.

IV. $\displaystyle\int u^{-1}\,du = \int \frac{du}{u} = \ln u + C.$

These formulas are proved by showing that the differential of the right member is equal to the expression under the integral sign. Thus, to prove III, we differentiate the right side and so obtain

$$d\left(\frac{u^{n+1}}{n+1} + C\right) = \frac{(n+1)\, u^n\, du}{n+1} = u^n\, du.$$

Formula I expresses that the integral of an algebraic sum of differentials is obtained by integrating them separately and adding the results. Integration is thus a distributive operation (Art. 16).

Formula II expresses that a constant factor can be transferred from one side of the symbol $\int$ to the other without changing the result. A variable cannot be transferred in this way. Thus it is not correct to write

$$\int x\, dx = x \int dx.$$

Example 1. $\int x^5\, dx.$

Apply formula III, letting $u = x$ and $n = 5$. Then $dx = du$ and

$$\int x^5\, dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C.$$

Example 2. $\int 3\sqrt{x}\, dx.$

By formula II we have

$$\int 3\sqrt{x}\, dx = 3\int x^{\frac{1}{2}}\, dx = \frac{3\, x^{\frac{3}{2}}}{\frac{3}{2}} + C = 2\, x^{\frac{3}{2}} + C.$$

Example 3. $\int (x-1)(x+2)\, dx.$

We expand and integrate term by term.

$$\int (x-1)(x+2)\, dx = \int (x^2 + x - 2)\, dx$$
$$= \tfrac{1}{3} x^3 + \tfrac{1}{2} x^2 - 2x + C.$$

Example 4. $\int \frac{x^2 - 2x + 1}{x^3} dx.$

Dividing by x^3 and using negative exponents, we get

$$\int \frac{x^2 - 2x + 1}{x^3} dx = \int (x^{-1} - 2x^{-2} + x^{-3})\, dx$$
$$= \ln x + 2x^{-1} - \tfrac{1}{2} x^{-2} + C$$
$$= \ln x + \frac{2}{x} - \frac{1}{2x^2} + C.$$

Example 5. $\int \sqrt{2x + 1}\, dx.$

If $u = 2x + 1$, $du = 2\, dx$. We therefore place a factor 2 before dx and $\frac{1}{2}$ outside the integral sign to compensate for it.

$$\int \sqrt{2x + 1}\, dx = \tfrac{1}{2} \int (2x + 1)^{\frac{1}{2}}\, 2\, dx = \tfrac{1}{2} \int u^{\frac{1}{2}}\, du$$
$$= \frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{3} (2x + 1)^{\frac{3}{2}} + C.$$

Example 6. $\int \frac{x\, dx}{x^2 + 1}.$

Apply IV with $u = x^2 + 1$. Then $du = 2x\, dx$ and

$$\int \frac{x\, dx}{x^2+1} = \frac{1}{2} \int \frac{2x\, dx}{x^2+1} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln u + C = \ln \sqrt{x^2 + 1} + C.$$

Example 7. $\int \frac{4x + 2}{2x - 1} dx.$

By division, we find

$$\frac{4x + 2}{2x - 1} = 2 + \frac{4}{2x - 1}.$$

Therefore

$$\int \frac{4x + 2}{2x - 1} dx = \int \left(2 + \frac{4}{2x - 1}\right) dx = 2x + 2 \ln(2x - 1) + C.$$

EXERCISES

Find the values of the following integrals:

1. $\int (x^3 - 4x^2 + 2x + 1)\, dx.$
2. $\int \left(x^2 - \frac{1}{x^2}\right) dx.$
3. $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx.$
4. $\int \left(\sqrt{2x} - \frac{1}{\sqrt{2x}}\right) dx.$
5. $\int x(x-1)^2\, dx.$
6. $\int (2x-1)^2\, dx.$
7. $\int \frac{2y+3}{y}\, dy.$
8. $\int \frac{(y+1)^2}{y^2}\, dy.$
9. $\int \frac{dx}{x+1}.$
10. $\int \frac{dx}{(x+1)^2}.$
11. $\int \frac{dx}{\sqrt{2x+1}}.$
12. $\int \frac{x\, dx}{x^2+2}.$
13. $\int \frac{x\, dx}{\sqrt{x^2-1}}.$
14. $\int \frac{x\, dx}{(a^2+x^2)^3}.$
15. $\int x\sqrt{a^2-x^2}\, dx.$
16. $\int \frac{x^2\, dx}{a^3+x^3}.$
17. $\int x^2 \sqrt{x^3-1}\, dx.$
18. $\int \frac{2x+1}{x^2+x+1}\, dx.$
19. $\int \frac{2x+a}{\sqrt{x^2+ax+b}}\, dx.$
20. $\int \frac{t^4\, dt}{1-at^5}.$
21. $\int t(a^2-t^2)^{\frac{3}{2}}\, dt.$
22. $\int \frac{x+1}{x-2}\, dx.$
23. $\int \left(2 + \frac{1}{2x^2+1}\right)\frac{x\, dx}{2x^2+1}.$
24. $\int \left(1 - \frac{1}{x}\right)^3 \frac{dx}{x^2}$
25. $\int \frac{x^{n-1}\, dx}{(x^n+a)^n}.$
26. $\int \sqrt{\frac{x+1}{x}}\, \frac{dx}{x^2}.$

71. Rates. — At time t, let the acceleration of a particle moving along a straight line be a, the velocity v, and the distance traversed s. Then

$$a = \frac{dv}{dt}, \qquad v = \frac{ds}{dt}.$$

Consequently

$$dv = a\, dt, \qquad ds = v\, dt.$$

If the acceleration is a known function of the time, integration of the first equation gives the velocity

$$v = \int a\,dt + C_1$$

Similarly, if the velocity is a known function of the time, integration of the second equation gives the distance traversed

$$s = \int v\,dt + C_2.$$

More generally, if the rate of change of any quantity z is a known function of the time,

$$\frac{dz}{dt} = f(t),$$

the value of z at time t is

$$z = \int f(t)\,dt + C.$$

If the value $z = z_0$ is known at some particular time $t = t_0$, by substituting

$$z = z_0, \qquad t = t_0$$

after integration, the constant C can be determined.

Example. A body falls from rest under the constant acceleration of gravity g. Find its velocity and the distance traversed as functions of the time t.

In this case

$$a = \frac{dv}{dt} = g.$$

Hence

$$v = \int g\,dt = gt + C.$$

Since the body starts from rest, $v = 0$ when $t = 0$. These values of v and t must satisfy the equation $v = gt + C$. Hence

$$0 = g \cdot 0 + C,$$

whence $C = 0$ and $v = gt$. Since $v = \frac{ds}{dt}$, $ds = gt\,dt$ and

$$s = \int gt\,dt + C = \tfrac{1}{2}\,gt^2 + C.$$

When $t = 0$, $s = 0$. Consequently, $C = 0$ and $s = \frac{1}{2}\,gt^2$.

72. Curves with a Given Slope. — If the slope of a curve is a given function of x,

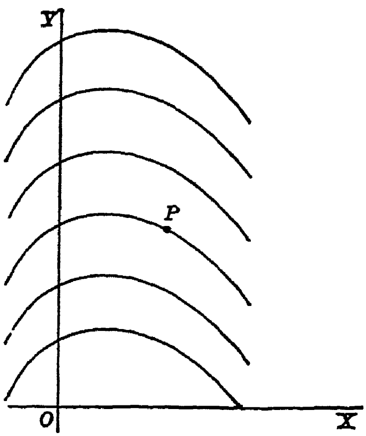

FIG. 72

$$\frac{dy}{dx} = f(x),$$

then

$$dy = f(x)\,dx$$

and

$$y = \int f(x)\,dx + C$$

is the equation of the curve.

Since the constant can have any value, there are an infinite number of curves having the given slope. If the curve is required to pass through a given point P, the value of C can be found by substituting the coördinates of P in the equation after integration.

Example 1. Find the curve passing through (1, 2) with slope equal to $2\,x$.

In this case

$$\frac{dy}{dx} = 2\,x.$$

Hence

$$y = \int 2\,x\,dx = x^2 + C.$$

Since the curve passes through (1, 2), the values $x = 1$, $y = 2$ must satisfy the equation, that is

$$2 = 1 + C.$$

Consequently, $C = 1$ and $y = x^2 + 1$ is the equation of the curve.

Example 2. On a certain curve

$$\frac{d^2y}{dx^2} = x.$$

If the curve passes through $(-2, 1)$ and has at that point the slope -2, find its equation.

By integration we get

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx = \int x\,dx = \tfrac{1}{2}\,x^2 + C.$$

At $(-2, 1)$, $x = -2$ and $\dfrac{dy}{dx} = -2$. Hence

$$-2 = 2 + C,$$

or $C = -4$. Consequently,

$$y = \int (\tfrac{1}{2}\,x^2 - 4)\,dx = \tfrac{1}{6}\,x^3 - 4\,x + C.$$

Since the curve passes through $(-2, 1)$,

$$1 = -\tfrac{8}{6} + 8 + C.$$

Consequently, $C = -5\tfrac{2}{3}$, and

$$y = \tfrac{1}{6}\,x^3 - 4\,x - 5\tfrac{2}{3}$$

is the equation of the curve.

73. Separation of the Variables. — The integration formulas contain only one variable. If a differential contains two or more variables, it must be reduced to a form in which each term contains a single variable. If this cannot be done, we cannot integrate the differential by our present methods.

Example 1. Find the curves such that the part of the tangent included between the coördinate axes is bisected at the point of tangency.

Let $P(x, y)$ be the point at which AB is tangent to the curve. Since P is the middle point of AB,

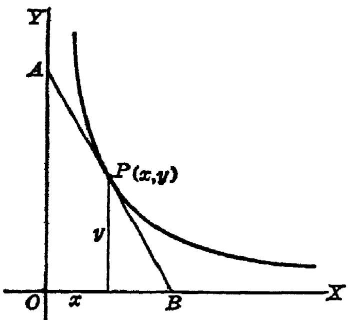

FIG. 73

$$OA = 2\,y, \qquad OB = 2\,x.$$

The slope of the curve at P is

$$\frac{dy}{dx} = -\frac{OA}{OB} = -\frac{y}{x}.$$

This can be written

$$\frac{dy}{y} + \frac{dx}{x} = 0.$$

Since each term contains a single variable, we can integrate and so get

$$\ln y + \ln x = C.$$

This is equivalent to

$$\ln xy = C.$$

Hence

$$xy = e^C = k.$$

C, and consequently k, can have any value. The curves are rectangular hyperbolas with the coördinate axes as asymptotes.

Example 2. According to Newton's law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - a),$$

where k is constant, a the temperature of the air, and θ the temperature at the time t of a body cooling in the air. Find θ as a function of t.

Multiplying by dt and dividing by $\theta - a$, Newton's equation becomes

$$\frac{d\theta}{\theta - a} = -k\,dt.$$

Integrating both sides, we get

$$\ln(\theta - a) = -kt + C.$$

Hence

$$\theta - a = e^{-kt+C} = e^C e^{-kt}.$$

When $t = 0$, let $\theta = \theta_0$. Then

$$\theta_0 - a = e^C e^0 = e^C,$$

and so

$$\theta - a = (\theta_0 - a)\,e^{-kt}$$

is the equation required.

Example 3. A cylindrical tank full of water has a leak at the bottom. Assuming that the water escapes at a rate proportional to the depth and that $\frac{1}{10}$ of it escapes the first day, how long will it take to half empty?

Let the radius of the tank be a, its height h and the depth of the water after t days x. The volume of the water at any time is $\pi a^2 x$ and its rate of change

$$-\pi a^2 \frac{dx}{dt}.$$

This is assumed to be proportional to x, that is,

$$\pi a^2 \frac{dx}{dt} = kx,$$

where k is constant. Separating the variables,

$$\frac{\pi a^2\,dx}{x} = k\,dt.$$

Integration gives

$$\pi a^2 \ln x = kt + C.$$

When $t = 0$ the tank is full and $x = h$. Hence

$$\pi a^2 \ln h = C.$$

Subtracting this from the preceding equation, we get

$$\pi a^2 \ln \frac{x}{h} = kt.$$

When $t = 1$, $x = \frac{9}{10} h$. Consequently,

$$\pi a^2 \ln \tfrac{9}{10} = k.$$

When $x = \frac{1}{2} h$,

$$t = \frac{\pi a^2 \ln \frac{x}{h}}{k} = \frac{\ln \frac{1}{2}}{\ln \frac{9}{10}} = 6.57 \text{ days.}$$

EXERCISES

1. A body, started vertically downward with a velocity of 30 ft./sec., falls under the acceleration of gravity. How far will it fall in t seconds?

2. From a point 60 ft. above the street a ball is thrown vertically upward with a speed of 100 ft./sec. Find its height above the street as a function of the time. Also find the highest point reached.

3. If the speed of a body at time t is

$$v = 2t + 3t^2,$$

find the distance traversed between $t = 2$ and $t = 5$.

4. A wheel starting from rest has a constant angular acceleration α. Find the angle through which it rotates in time t.

5. A rotating wheel is brought to rest by the action of friction which causes a constant negative angular acceleration $-k$. If the initial angular velocity is ω_0 find the angle turned through in coming to rest.

6. Find the equation of the curve with slope $1 - x$ passing through the point $(1, 0)$.

7. On a certain curve

$$\frac{d^2y}{dx^2} = x - 1.$$

If the curve passes through $(-1, 1)$ and has at that point the slope 2, find its equation.

8. Every tangent to a curve bisects the portion of the x-axis between the origin and the ordinate at its point of contact. If the curve passes through the point (2, 4), find its equation.

9. The portion of the tangent to a curve between the x-axis and the point of tangency is bisected by the y-axis. If the curve passes through the point (1, 2), find its equation.

10. When bacteria grow in the presence of unlimited food they increase at a rate proportional to the number present. Express that number as a function of the time.

11. Many chemical transformations occur at a rate proportional to the amount x of the substance still untransformed. Show that $x = ce^{-kt}$. What does c represent?

12. The rate at which water flows from a small opening at the bottom of a tank is proportional to the square root of the depth of the water. If half the water flows from a cylindrical tank (with vertical axis) in 5 minutes, find the time required to empty.

CHAPTER XI

FORMULAS AND METHODS OF INTEGRATION

74. Formulas. — The following is a short list of integration formulas. In these u is any variable or function of a single variable and du is its differential. The constant is omitted but it should be added to each function determined by integration. A more extended list of formulas is given in the Appendix.

I. $\int u^n\,du = \frac{u^{n+1}}{n+1}$, if n is not -1.

II. $\int \frac{du}{u} = \ln u.$

III. $\int \cos u\,du = \sin u.$

IV. $\int \sin u\,du = -\cos u.$

V. $\int \sec^2 u\,du = \tan u.$

VI. $\int \csc^2 u\,du = -\cot u.$

VII. $\int \sec u \tan u\,du = \sec u.$

VIII. $\int \csc u \cot u\,du = -\csc u.$

IX. $\int \tan u\,du = -\ln \cos u.$

X. $\int \cot u\,du = \ln \sin u.$

XI. $\int \sec u\,du = \ln(\sec u + \tan u).$

XII. $\int \csc u\,du = \ln(\csc u - \cot u).$

XIII. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a}.$ *

XIV. $\displaystyle\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}.$

XV. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a}.$ *

XVI. $\displaystyle\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right).$

XVII. $\displaystyle\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\frac{u - a}{u + a}.$

XVIII. $\displaystyle\int e^u\, du = e^u.$

Any one of these formulas can be proved by showing that the differential of the right member is equal to the expression under the integral sign. Thus to show that

$$\int \sec u\, du = \ln(\sec u + \tan u),$$

we note that

$$d\ln(\sec u + \tan u) = \frac{(\sec u \tan u + \sec^2 u)\, du}{\sec u + \tan u} = \sec u\, du.$$

75. Integration by Substitution. — When some function of the variable is taken as u, a given differential may assume the form of the differential in one of the integration formulas or differ from such form only by a constant factor. Integration accomplished in this way is called integration by substitution.

Each differential is the product of a function of u by du. More errors result from failing to pay attention to the du

* In formulas XIII and XV it is assumed that $\sin^{-1}\frac{u}{a}$ is an angle in the 1st or 4th quadrant, and $\sec^{-1}\frac{u}{a}$ an angle in the 1st or 2nd quadrant. In other cases the algebraic sign of the result must be changed.

than from any other one cause. Thus the student may carelessly conclude from formula III that the integral of a cosine is a sine and so write

$$\int \cos 2\,x\,dx = \sin 2\,x.$$

If, however, we let $2\,x = u$, dx is not du but $\frac{1}{2}\,du$ and so

$$\int \cos 2\,x\,dx = \tfrac{1}{2}\int \cos u\,du = \tfrac{1}{2}\sin u = \tfrac{1}{2}\sin 2\,x.$$

Example 1. $\int \sin^3 x \cos x\,dx.$

If we let $u = \sin x$, $du = \cos x\,dx$ and

$$\int \sin^3 x \cos x\,dx = \int u^3\,du = \tfrac{1}{4}\,u^4 + C = \tfrac{1}{4}\sin^4 x + C.$$

Example 2. $\int \frac{\sin \frac{1}{3}\,x\,dx}{1 + \cos \frac{1}{3}\,x}.$

We observe that $\sin \frac{1}{3}\,x\,dx$ differs only by a constant factor from the differential of $1 + \cos \frac{1}{3}\,x$. Hence we let

$$u = 1 + \cos \tfrac{1}{3}\,x.$$

Then $\qquad du = -\frac{1}{3}\sin \frac{1}{3}\,x\,dx, \qquad \sin \frac{1}{3}\,x\,dx = -3\,du,$

and

$$\int \frac{\sin \frac{1}{3}\,x\,dx}{1 + \cos \frac{1}{3}\,x} = -3\int \frac{du}{u} = -3 \ln u + C$$
$$= -3 \ln (1 + \cos \tfrac{1}{3}\,x) + C.$$

Example 3. $\int (\tan x + \sec x) \sec x\,dx.$

Expanding we get

$$\int (\tan x + \sec x) \sec x\,dx = \int \tan x \sec x\,dx + \int \sec^2 x\,dx$$
$$= \sec x + \tan x + C.$$

Example 4. $\displaystyle\int \frac{3\,dx}{\sqrt{2 - 3x^2}}$.

This resembles the integral in formula XIII. Let $u = x\sqrt{3}$, $a = \sqrt{2}$. Then $du = \sqrt{3}\,dx$ and

$$\int \frac{3\,dx}{\sqrt{2 - 3x^2}} = \int \frac{3\dfrac{du}{\sqrt{3}}}{\sqrt{a^2 - u^2}} = \sqrt{3}\int \frac{du}{\sqrt{a^2 - u^2}}$$

$$= \sqrt{3}\sin^{-1}\frac{u}{a} + C = \sqrt{3}\sin^{-1}\frac{x\sqrt{3}}{\sqrt{2}} + C.$$

Example 5. $\displaystyle\int \frac{dt}{t\sqrt{4t^2 - 9}}$.

This suggests the integral in formula XV. Let $u = 2t$, $a = 3$. Then

$$\int \frac{dt}{t\sqrt{4t^2 - 9}} = \int \frac{2\,dt}{2t\sqrt{4t^2 - 9}} = \int \frac{du}{u\sqrt{u^2 - a^2}}$$

$$= \frac{1}{a}\sec^{-1}\frac{u}{a} + C = \frac{1}{3}\sec^{-1}\frac{2t}{3} + C.$$

Example 6. $\displaystyle\int \frac{x\,dx}{\sqrt{2x^2 + 1}}$.

This may suggest formula XVI. If, however, we let $u = x\sqrt{2}$, $du = \sqrt{2}\,dx$, which is not a constant times $x\,dx$. We should let

$$u = 2x^2 + 1.$$

Then $x\,dx = \frac{1}{4}\,du$ and

$$\int \frac{x\,dx}{\sqrt{2x^2 + 1}} = \frac{1}{4}\int \frac{du}{\sqrt{u}} = \frac{1}{4}\int u^{-\frac{1}{2}}\,du$$

$$= \tfrac{1}{2}\sqrt{u} + C = \tfrac{1}{2}\sqrt{2x^2 + 1} + C.$$

Example 7. $\displaystyle\int e^{\tan x}\sec^2 x\,dx$.

If $u = \tan x$, by formula XVIII

$$\int e^{\tan x}\sec^2 x\,dx = \int e^u\,du = e^u + C = e^{\tan x} + C.$$

EXERCISES

Determine the values of the following integrals:

1. $\int (\cos 3x + \sin 2x)\,dx.$
2. $\int \sin\left(\frac{3-2x}{5}\right)dx.$
3. $\int \sec^2 \frac{1}{3}\theta\,d\theta.$
4. $\int \csc \frac{1}{2}\theta \cot \frac{1}{2}\theta\,d\theta.$
5. $\int \cos\theta \sin\theta\,d\theta.$
6. $\int \frac{dx}{\cos^2 x}.$
7. $\int \frac{dx}{\sin^2 2x}.$
8. $\int \frac{\cos x\,dx}{\sin^2 x}.$
9. $\int \frac{\sin x\,dx}{\cos^4 x}.$
10. $\int \frac{\cos x\,dx}{1+\sin x}.$
11. $\int (\csc\theta - \cot\theta)\csc\theta\,d\theta.$
12. $\int x\sin(x^2+1)\,dx.$
13. $\int \frac{1+\sin 2x}{\cos^2 2x}\,dx.$
14. $\int (1+\sec x)^2\,dx.$
15. $\int \frac{(\cos x - \sin x)^2}{\sin x}\,dx.$
16. $\int \sin^3 x \cos x\,dx.$
17. $\int \tan^2 x \sec^2 x\,dx.$
18. $\int \tan x \sec^3 x\,dx.$
19. $\int \sin x \cos^4 x\,dx.$
20. $\int \frac{\csc^2 x\,dx}{1+2\cot x}.$
21. $\int \frac{dx}{\sqrt{4-x^2}}.$
22. $\int \frac{dx}{\sqrt{3-4x^2}}.$
23. $\int \frac{2\,dx}{4x^2+3}.$
24. $\int \frac{dx}{x\sqrt{2x^2-3}}.$
25. $\int \frac{dy}{4y^2+3}.$
26. $\int \frac{dt}{\sqrt{9t^2+1}}.$
27. $\int \frac{2\,dx}{x\sqrt{4x^2-9}}.$
28. $\int \frac{2x+3}{\sqrt{4-x^2}}\,dx.$
29. $\int \frac{x+1}{\sqrt{x^2+4}}\,dx.$
30. $\int \frac{\cos x\,dx}{\sqrt{1+\sin^2 x}}.$
31. $\int \frac{\sin x \cos x\,dx}{\sqrt{1+\sin^2 x}}.$
32. $\int \frac{\cos x\,dx}{1+\sin^2 x}.$
33. $\int \frac{\sin\theta\,d\theta}{\sqrt{1-\cos\theta}}.$
34. $\int \frac{dx}{x(1+\ln x)}.$
35. $\int \frac{x\,dx}{\sqrt{a^4-x^4}}.$
36. $\int xe^{-x^2}\,dx.$
37. $\int (e^{ax}-e^{-ax})^2\,dx.$
38. $\int \frac{e^{2x}\,dx}{1+e^{2x}}.$
39. $\int \frac{e^x\,dx}{1+e^{2x}}.$
40. $\int \frac{dt}{\sqrt{e^{2t}-1}}.$

76. Integrals Containing $ax^2 + bx + c$. — Integrals containing a quadratic expression $ax^2 + bx + c$ can often be reduced to manageable form by completing the square of $ax^2 + bx$.

Example 1. $\displaystyle\int \frac{dx}{3x^2 + 6x + 5}$.

Completing the square, we get

$3x^2 + 6x + 5 = 3(x^2 + 2x + 1) + 2 = 3(x + 1)^2 + 2.$

If then $u = (x + 1)\sqrt{3}$,

$$\int \frac{dx}{3x^2 + 6x + 5} = \int \frac{d(x+1)}{3(x+1)^2 + 2} = \frac{1}{\sqrt{3}}\int \frac{du}{u^2 + 2}$$
$$= \frac{1}{\sqrt{6}} \tan^{-1} \frac{(x+1)\sqrt{3}}{\sqrt{2}} + C.$$

Example 2. $\displaystyle\int \frac{2\,dx}{\sqrt{2 - 3x - x^2}}$.

The coefficient of x^2 being negative, we place the terms x^2 and $3x$ in a parenthesis preceded by a minus sign. Thus

$$2 - 3x - x^2 = 2 - (x^2 + 3x) = \tfrac{17}{4} - (x + \tfrac{3}{2})^2.$$

If then, $u = x + \frac{3}{2}$, we have

$$\int \frac{2\,dx}{\sqrt{2 - 3x - x^2}} = 2\int \frac{du}{\sqrt{\frac{17}{4} - u^2}} = 2 \sin^{-1} \frac{x + \frac{3}{2}}{\frac{1}{2}\sqrt{17}} + C.$$

Example 3. $\displaystyle\int \frac{(2x - 1)\,dx}{\sqrt{4x^2 + 4x + 2}}$.

Since the numerator contains the first power of x, we resolve the integral into two parts,

$$\int \frac{(2x-1)\,dx}{\sqrt{4x^2+4x+2}} = \frac{1}{4}\int \frac{(8x+4)\,dx}{\sqrt{4x^2+4x+2}} - 2\int \frac{dx}{\sqrt{4x^2+4x+2}}.$$

In the first integral on the right the numerator is taken equal to the differential of $4x^2 + 4x + 2$. In the second the numerator is dx. The outside factors $\frac{1}{4}$ and -2 are chosen

so that the two sides of the equation are equal. The first integral has the form

$$\frac{1}{4}\int\frac{du}{\sqrt{u}}=\frac{1}{2}\sqrt{u}=\frac{1}{2}\sqrt{4\,x^2+4\,x+2}.$$

The second integral is evaluated by completing the square. The final result is

$$\int\frac{(2\,x-1)\,dx}{\sqrt{4\,x^2+4\,x+2}}=\frac{1}{2}\sqrt{4\,x^2+4\,x+2}$$
$$-\ln\left(2\,x+1+\sqrt{4\,x^2+4\,x+2}\right)+C.$$

EXERCISES

1. $\displaystyle\int\frac{dx}{x^2+6\,x+13}$.

2. $\displaystyle\int\frac{dx}{\sqrt{3+4\,x-4\,x^2}}$.

3. $\displaystyle\int\frac{dx}{\sqrt{3\,x^2+6\,x+2}}$.

4. $\displaystyle\int\frac{dx}{\sqrt{2+6\,x-3\,x^2}}$.

5. $\displaystyle\int\frac{dx}{(x-1)\sqrt{x^2-2\,x-3}}$.

6. $\displaystyle\int\frac{dx}{(2\,x-1)\sqrt{x^2-x}}$.

7. $\displaystyle\int\frac{(x-1)\,dx}{4\,x^2-4\,x+2}$.

8. $\displaystyle\int\frac{(2\,x-1)\,dx}{\sqrt{3\,x^2-6\,x-1}}$.

9. $\displaystyle\int\frac{x\,dx}{x^2+2\,x+2}$.

10. $\displaystyle\int\frac{(x+1)\,dx}{(2\,x+1)\sqrt{4\,x^2+4\,x-1}}$.

11. $\displaystyle\int\frac{(3\,x-3)\,dx}{(x^2-2\,x+3)^{\frac{3}{2}}}$.

12. $\displaystyle\int\frac{e^x\,dx}{e^{2x}+2\,e^x+3}$.

77. Integrals of Trigonometric Functions. — A power of a trigonometric function multiplied by its differential can be integrated by formula I. Thus, if $u=\tan x$,

$$\int\tan^4 x\cdot\sec^2 x\,dx=\int u^4\,du=\tfrac{1}{5}\tan^5 x+C.$$

Differentials can often be reduced to the above form by trigonometric transformations. This is illustrated by the following examples.

Example 1. $\int \sin^4 x \cos^3 x \, dx.$

If we take $\cos x \, dx$ as du and use the relation $\cos^2 x = 1 - \sin^2 x$, the other factors can be expressed in terms of $\sin x$ *without introducing radicals.* Thus

$$\int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \cos^2 x \cdot \cos x \, dx$$

$$= \int \sin^4 x \, (1 - \sin^2 x) \, d \sin x = \tfrac{1}{5} \sin^5 x - \tfrac{1}{7} \sin^7 x + C.$$

Example 2. $\int \tan^3 x \sec^4 x \, dx.$

If we take $\sec^2 x \, dx$ as du and use the relation $\sec^2 x = 1 + \tan^2 x$, the other factors can be expressed in terms of $u = \tan x$ without introducing radicals. Thus

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \cdot \sec^2 x \cdot \sec^2 x \, dx$$

$$= \int \tan^3 x \, (1 + \tan^2 x) \, d \tan x$$

$$= \tfrac{1}{4} \tan^4 x + \tfrac{1}{6} \tan^6 x + C.$$

Example 3. $\int \tan^3 x \sec^3 x \, dx.$

If we take $\tan x \sec x \, dx = d \sec x$ as du, and use the relation $\tan^2 x = \sec^2 x - 1$, the integral takes the form

$$\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \cdot \sec^2 x \cdot \tan x \sec x \, dx$$

$$= \int (\sec^2 x - 1) \sec^2 x \cdot d \sec x$$

$$= \tfrac{1}{5} \sec^5 x - \tfrac{1}{3} \sec^3 x + C.$$

Example 4. $\int \sin 2 x \cos 3 x \, dx.$

This is the product of the sine of one angle and the cosine of another. This product can be resolved into a sum or

difference by the formula

$$\sin A \cos B = \tfrac{1}{2} [\sin (A + B) + \sin (A - B)].$$

Thus

$$\begin{aligned} \sin 2x \cos 3x &= \tfrac{1}{2} [\sin 5x + \sin (-x)] \\ &= \tfrac{1}{2} [\sin 5x - \sin x] \end{aligned}$$

Consequently,

$$\begin{aligned} \int \sin 2x \cos 3x \, dx &= \tfrac{1}{2} \int (\sin 5x - \sin x) \, dx \\ &= -\tfrac{1}{10} \cos 5x + \tfrac{1}{2} \cos x + C. \end{aligned}$$

Example 5. $\int \tan^5 x \, dx.$

If we replace $\tan^2 x$ by $\sec^2 x - 1$, the integral becomes

$$\int \tan^5 x \, dx = \int \tan^3 x (\sec^2 x - 1) \, dx = \tfrac{1}{4} \tan^4 x - \int \tan^3 x \, dx.$$

The integral is thus made to depend on a simpler one

$\int \tan^3 x \, dx.$ Similarly,

$$\int \tan^3 x \, dx = \int \tan x (\sec^2 x - 1) \, dx = \tfrac{1}{2} \tan^2 x + \ln \cos x.$$

Hence finally

$$\int \tan^5 x \, dx = \tfrac{1}{4} \tan^4 x - \tfrac{1}{2} \tan^2 x - \ln \cos x + C.$$

78. Even Powers of Sines and Cosines. — Integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

where m or n is odd can be evaluated by the methods of Art. 77. If both m and n are even, however, those methods

fail. In that case we can evaluate the integral by the use of the formulas

$$\left.\begin{aligned} \sin^2 u &= \frac{1-\cos 2u}{2}, \\ \cos^2 u &= \frac{1+\cos 2u}{2}, \\ \sin u \cos u &= \frac{\sin 2u}{2}. \end{aligned}\right\} \tag{11}$$

Example 1. $\int \cos^4 x \, dx.$

By the above formulas

$$\begin{aligned} \int \cos^4 x \, dx &= \int (\cos^2 x)^2 \, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \int \left(\tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\cos^2 2x\right) dx \\ &= \int \left[\tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}(1+\cos 4x)\right] dx \\ &= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C. \end{aligned}$$

Example 2. $\int \cos^2 x \sin^2 x \, dx.$

$$\begin{aligned} \int \cos^2 x \sin^2 x \, dx &= \int \tfrac{1}{4}\sin^2 2x \, dx = \int \tfrac{1}{8}(1 - \cos 4x) \, dx \\ &= \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C. \end{aligned}$$

79. Trigonometric Substitutions. — If a differential contains $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$, one should first try this radical as a new variable. If that substitution fails, the differential can often be integrated by taking x, a, and the square root as the three sides of a right triangle and using one of its acute angles as new variable.

Example 1. $\int x^3 \sqrt{a^2 - x^2} \, dx.$

Let $\sqrt{a^2 - x^2} = z.$ Then

$$x^2 = a^2 - z^2, \quad x\,dx = -z\,dz.$$

whence

$$\begin{aligned}\int x^3\sqrt{a^2-x^2}\,dx &= \int x^2\sqrt{a^2-x^2}\cdot x\,dx.\\ &= \int (a^2-z^2)\,z\,(-z\,dz)\\ &= \frac{z^5}{5}-\frac{a^2z^3}{3}+C.\\ &= -\tfrac{1}{15}\,(3\,x^2+2\,a^2)\,(a^2-x^2)^{\frac{3}{2}}+C.\end{aligned}$$

Example 2. $\displaystyle\int\sqrt{a^2-x^2}\,dx.$

Take a as hypotenuse and x as one side of a right triangle (Fig. 79a). Then

$$x = a\sin\theta,\qquad dx = a\cos\theta\,d\theta,\qquad \sqrt{a^2-x^2} = a\cos\theta,$$

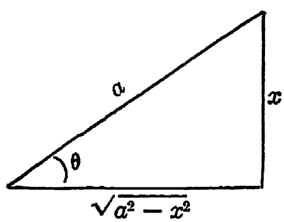

FIG. 79a.

and

$$\begin{aligned}\int\sqrt{a^2-x^2}\,dx &= \int a^2\cos^2\theta\,d\theta = \int\frac{a^2}{2}(1+\cos 2\,\theta)\,d\theta\\ &= \frac{a^2}{2}\left[\theta+\tfrac{1}{2}\sin 2\,\theta\right]+C\\ &= \frac{a^2}{2}\left[\theta+\sin\theta\cos\theta\right]+C.\end{aligned}$$

From the triangle

$$\theta = \sin^{-1}\frac{x}{a},\qquad \sin\theta = \frac{x}{a},\qquad \cos\theta = \frac{\sqrt{a^2-x^2}}{a}.$$

Hence

$$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C.$$

Example 3. $\displaystyle\int \frac{dx}{(x^2 + a^2)^2}$.

Take a and x as the two sides of a right triangle (Fig. 79b). Then

$$x = a \tan \theta, \qquad dx = a \sec^2 \theta \, d\theta$$

and

$$a^2 + x^2 = a^2 \sec^2 \theta.$$

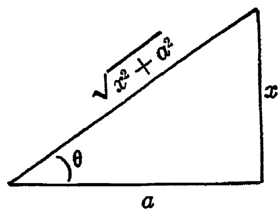

FIG. 79b.

Consequently

$$\int \frac{dx}{(x^2 + a^2)^2} = \frac{1}{a^3} \int \frac{d\theta}{\sec^2 \theta} = \frac{1}{a^3} \int \cos^2 \theta \, d\theta$$
$$= \frac{1}{2\,a^3} (\theta + \sin \theta \cos \theta) + C$$
$$= \frac{1}{2\,a^3} \left[\tan^{-1} \frac{x}{a} + \frac{ax}{a^2 + x^2} \right] + C.$$

EXERCISES

1. $\int \sin^3 x \, dx.$
2. $\int \cos^5 x \, dx.$
3. $\int \sin^2 x \cos^3 x \, dx.$
4. $\int \sin^3 3\,\theta \cos^3 3\,\theta \, d\theta.$
5. $\int (\cos^2 \theta - \sin^2 \theta) \sin \theta \, d\theta.$
6. $\displaystyle\int \frac{\sin^3 x \, dx}{1 - \cos x}.$

7. $\displaystyle\int \frac{\cos^2 x\, dx}{\sin x}$.

8. $\displaystyle\int \frac{\cos^3 x\, dx}{\sin x}$.

9. $\displaystyle\int \sec^4 x\, dx$.

10. $\displaystyle\int \csc^6 y\, dy$.

11. $\displaystyle\int \tan^2 x\, dx$.

12. $\displaystyle\int \tan \tfrac{1}{2} x \sec^3 \tfrac{1}{2} x\, dx$.

13. $\displaystyle\int \tan^5 x \sec^3 x\, dx$.

14. $\displaystyle\int \cot^3 x\, dx$.

15. $\displaystyle\int \frac{\cos^2 x\, dx}{\sin^6 x}$.

16. $\displaystyle\int \sec x \csc x\, dx$.

17. $\displaystyle\int \cos x \sin 2x\, dx$.

18. $\displaystyle\int \sin x \cos 3x\, dx$.

19. $\displaystyle\int \sin 2x \sin 3x\, dx$.

20. $\displaystyle\int \cos 2x \cos 4x\, dx$.

21. $\displaystyle\int \sin^2 (2x)\, dx$.

22. $\displaystyle\int \cos^2 (4x)\, dx$.

23. $\displaystyle\int \cos^2 x \sin^4 x\, dx$.

24. $\displaystyle\int \cos^4 \tfrac{1}{2} x \sin^2 \tfrac{1}{2} x\, dx$.

25. $\displaystyle\int \sin^6 x\, dx$.

26. $\displaystyle\int \frac{dx}{1 - \cos x}$.

27. $\displaystyle\int \frac{dx}{1 + \sin x}$.

28. $\displaystyle\int \sqrt{1 + \cos \theta}\, d\theta$.

29. $\displaystyle\int x\sqrt{x^2 - a^2}\, dx$.

30. $\displaystyle\int \sqrt{x^2 - a^2}\, dx$.

31. $\displaystyle\int x^3\sqrt{x^2 + a^2}\, dx$.

32. $\displaystyle\int \sqrt{x^2 + a^2}\, dx$.

33. $\displaystyle\int \frac{x^2\, dx}{\sqrt{x^2 + a^2}}$.

34. $\displaystyle\int \frac{dx}{(x^2 - a^2)^{\frac{3}{2}}}$.

35. $\displaystyle\int \frac{dx}{x\sqrt{a^2 - x^2}}$.

36. $\displaystyle\int \frac{x\, dx}{(a^2 - x^2)^{\frac{3}{2}}}$.

80. Integration of Rational Fractions. — A fraction, such as

$$\frac{x^3 + 3x}{x^2 - 2x - 3},$$

whose numerator and denominator are polynomials, is called a *rational fraction.*

If the degree of the numerator is equal to or greater than that of the denominator, the fraction should be reduced by division. Thus

$$\frac{x^3 + 3x}{x^2 - 2x - 3} = x + 2 + \frac{10x + 6}{x^2 - 2x - 3}.$$

A fraction with numerator of lower degree than its denominator can be resolved into a sum of *partial fractions* with denominators that are factors of the original denominator. Thus

$$\frac{10x+6}{x^2-2x-3}=\frac{10x+6}{(x-3)(x+1)}=\frac{9}{x-3}+\frac{1}{x+1}.$$

These fractions can often be found by trial. If not, proceed as in the following examples.

CASE 1. *Factors of the denominator all of the first degree and none repeated.*

Example 1. $\displaystyle\int\frac{x^4+2x+6}{x^3+x^2-2x}dx.$

Dividing numerator by denominator, we get

$$\frac{x^4+2x+6}{x^3+x^2-2x}=x-1+\frac{3x^2+6}{x^3+x^2-2x}$$
$$=x-1+\frac{3x^2+6}{x(x-1)(x+2)}.$$

Assume

$$\frac{3x^2+6}{x(x-1)(x+2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}.$$

The two sides of this equation are merely different ways of writing the same function. If then we clear of fractions, the two sides of the resulting equation

$$\begin{aligned}3x^2+6&=A(x-1)(x+2)+Bx(x+2)+Cx(x-1)\\&=(A+B+C)x^2+(A+2B-C)x-2A\end{aligned}$$

are identical. That is

$$A+B+C=3,\quad A+2B-C=0,\quad -2A=6.$$

Solving these equations, we get

$$A=-3,\qquad B=3,\qquad C=3.$$

Conversely, if A, B, C have these values, the above equations are identically satisfied. Therefore

$$\int \frac{x^4+2x+6}{x^3+x^2-2x}\,dx = \int \left(x - 1 - \frac{3}{x} + \frac{3}{x-1} + \frac{3}{x+2}\right) dx$$

$$= \tfrac{1}{2}x^2 - x - 3\ln x + 3\ln(x-1) + 3\ln(x+2) + C$$

$$= \frac{1}{2}x^2 - x + 3\ln\frac{(x-1)(x+2)}{x} + C.$$

The constants can often be determined more easily by substituting particular values for x on the two sides of the equation. Thus, the equation above,

$$3x^2 + 6 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1),$$

is an identity, that is, it is satisfied by all values of x. In particular, if $x = 0$, it becomes

$$6 = -2A,$$

whence $A = -3$. Similarly, by substituting $x = 1$ and $x = -2$, we get

$$9 = 3B, \quad 18 = 6C,$$

whence $\quad B = 3, \quad C = 3.$

CASE 2. *Factors of the denominator all of first degree but some repeated.*

Example 2. $\displaystyle\int \frac{(8x^3+7)\,dx}{(x+1)(2x+1)^3}.$

Assume

$$\frac{8x^3+7}{(x+1)(2x+1)^3} = \frac{A}{x+1} + \frac{B}{(2x+1)^3} + \frac{C}{(2x+1)^2} + \frac{D}{2x+1}.$$

Corresponding to the repeated factor $(2x+1)^3$, we thus introduce fractions with $(2x+1)^3$ and all lower powers as

denominators. Clearing and solving as before, we find

$$A = 1, \quad B = 12, \quad C = -6, \quad D = 0.$$

Hence

$$\int \frac{8x^3+7}{(x+1)(2x+1)^3}dx = \int\left[\frac{1}{x+1} + \frac{12}{(2x+1)^3} - \frac{6}{(2x+1)^2}\right]dx$$
$$= \ln(x+1) - \frac{3}{(2x+1)^2} + \frac{3}{2x+1} + C.$$

CASE 3. *Denominator containing factors of the second degree but none repeated.*

Example 3. $\displaystyle\int \frac{4x^2+x+1}{x^3-1}dx.$

The factors of the denominator are $x - 1$ and $x^2 + x + 1$. Assume

$$\frac{4x^2+x+1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

With the quadratic denominator $x^2 + x + 1$, we thus use a numerator that is not a single constant but a linear function $Bx + C$. Clearing fractions and solving for A, B, C, we find

$$A = 2, \quad B = 2, \quad C = 1.$$

Therefore

$$\int \frac{4x^2+x+1}{x^3-1}dx = \int\left(\frac{2}{x-1} + \frac{2x+1}{x^2+x+1}\right)dx$$
$$= 2\ln(x-1) + \ln(x^2+x+1) + C.$$

CASE 4. *Denominator containing factors of the second degree, some being repeated.*

Example 4. $\displaystyle\int \frac{x^3+1}{x(x^2+1)^2}dx.$

Assume

$$\frac{x^3+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)^2} + \frac{Dx+E}{x^2+1}.$$

Corresponding to the repeated second degree factor $(x^2 + 1)^2$, we introduce partial fractions having as denominators $(x^2 + 1)^2$ and all lower powers of $x^2 + 1$, the numerators being all of first degree. Clearing fractions and solving for A, B, C, D, E, we find

$$A = 1, \quad B = -1, \quad C = -1, \quad D = -1, \quad E = 1.$$

Hence

$$\int \frac{x^3 + 1}{x(x^2 + 1)^2} dx = \int \left[\frac{1}{x} - \frac{x + 1}{(x^2 + 1)^2} - \frac{x - 1}{x^2 + 1}\right] dx$$

$$= \ln \frac{x}{\sqrt{x^2 + 1}} + \frac{1}{2} \tan^{-1} x - \frac{x - 1}{2\,(x^2 + 1)} + C.$$

81. Integrals Containing $(ax + b)^{\frac{p}{q}}$.—Integrals containing $(ax + b)^{\frac{p}{q}}$ can be rationalized by the substitution

$$ax + b = z^q.$$

If several fractional powers of the same linear function $ax + b$ occur, the substitution

$$ax + b = z^n$$

may be used, n being so chosen that all the roots can be extracted.

Example 1. $\displaystyle\int \frac{dx}{1 + \sqrt{x}}.$

Let $x = z^2$. Then $dx = 2\,z\,dz$ and

$$\int \frac{dx}{1 + \sqrt{x}} = \int \frac{2\,z\,dz}{1 + z} = \int \left(2 - \frac{2}{1 + z}\right) dz$$

$$= 2\,z - 2 \ln (1 + z) + C$$

$$= 2\sqrt{x} - 2 \ln (1 + \sqrt{x}) + C.$$

EXERCISES

1. $\int \frac{x^3 + x^2}{x^2 - 3x + 2} dx.$
2. $\int \frac{x + 2}{x^2 + x} dx.$
3. $\int \frac{dx}{x(x^2 - 1)}.$
4. $\int \frac{x^3 + 4}{x^3 - 4x} dx.$
5. $\int \frac{x^3 + 1}{x^3 - x^2} dx.$
6. $\int \frac{x\, dx}{(x + 1)^2}.$
7. $\int \frac{dx}{(x^2 - 1)^2}.$
8. $\int \frac{2x^2 + 2x - 4}{x^4 - 2x^3} dx.$
9. $\int \frac{x + 1}{x(x^2 + 1)} dx.$
10. $\int \frac{3(x + 1)}{x^3 - 1} dx.$
11. $\int \frac{x\, dx}{x^4 - 4}.$
12. $\int \frac{x^3 + 2x^2}{(x + 1)(x^2 + 2x + 2)} dx.$
13. $\int \frac{x^3\, dx}{(x^2 + 4)^2}.$
14. $\int \frac{1 + \sqrt{x}}{1 - \sqrt{x}} dx.$
15. $\int \frac{x\, dx}{\sqrt{x + 1}}.$
16. $\int x\sqrt{x - a}\, dx.$
17. $\int \frac{\sqrt{x + 2}}{x + 3} dx.$
18. $\int \frac{x^{\frac{1}{2}}\, dx}{1 + x^{\frac{1}{3}}}.$

82. Integration by Parts. — From the formula

$$d(uv) = u\, dv + v\, du$$

we get

$$u\, dv = d(uv) - v\, du,$$

whence

$$\int u\, dv = uv - \int v\, du. \tag{82}$$

If $\int v\, du$ is known this gives $\int u\, dv$. Integration by the use of this formula is called *integration by parts*.

Example 1. $\int \ln x\, dx.$

Let $u = \ln x$, $dv = dx$. Then $du = \frac{dx}{x}$, $v = x$, and

$$\int \ln x\, dx = \ln x \cdot x - \int x \cdot \frac{dx}{x}$$
$$= x(\ln x - 1) + C.$$

Example 2. $\int x^2 \sin x \, dx.$

Let $u = x^2$ and $dv = \sin x \, dx$. Then $du = 2\,x\,dx$, $v = -\cos x$, and

$$\int x^2 \sin x \, dx = -x^2 \cos x + \int 2\,x \cos x \, dx.$$

A second integration by parts with $u = 2\,x$, $dv = \cos x \, dx$ gives

$$\int 2\,x \cos x \, dx = 2\,x \sin x - \int 2 \sin x \, dx$$
$$= 2\,x \sin x + 2 \cos x + C.$$

Hence finally

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2\,x \sin x + 2 \cos x + C.$$

The method of integration by parts applies particularly to functions that are simplified by differentiation, like $\ln x$, or to products of functions of different classes, like $x \sin x$. In applying the method the given differential must be resolved into a product $u \cdot dv$. The part called dv must have a known integral and the part called u should usually be simplified by differentiation.

Sometimes after integration by parts a multiple of the original differential appears on the right side of the equation. It can be transposed to the other side and the integral can be solved for algebraically. This is shown in the following examples.

Example 3. $\int \sqrt{a^2 - x^2}\, dx.$

Integrating by parts with $u = \sqrt{a^2 - x^2}$, $dv = dx$, we get

$$\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} - \int \frac{-x^2\, dx}{\sqrt{a^2 - x^2}}.$$

Adding a^2 to the numerator of the integral and subtracting an equivalent integral, this becomes

$$\int \sqrt{a^2 - x^2}\,dx = x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}dx + a^2\int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\,dx + a^2\int \frac{dx}{\sqrt{a^2 - x^2}}.$$

Transposing $\int \sqrt{a^2 - x^2}\,dx$ and dividing by 2, we get

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$$

Example 4. $\int e^{ax}\cos bx\,dx.$

Integrating by parts with $u = e^{ax}$, $dv = \cos bx\,dx$, we get

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\int e^{ax}\sin bx\,dx.$$

Integrating by parts again with $u = e^{ax}$, $dv = \sin bx\,dx$, this becomes

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\left[-\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx\,dx\right]$$

$$= e^{ax}\left(\frac{b\sin bx + a\cos bx}{b^2}\right) - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx.$$

Transposing the last integral and dividing by $1 + \frac{a^2}{b^2}$, this gives

$$\int e^{ax}\cos bx\,dx = e^{ax}\left(\frac{b\sin bx + a\cos bx}{a^2 + b^2}\right).$$

83. Reduction Formulas. — Integration by parts is often used to make an integral depend on a simpler one and so to obtain a formula by repeated application of which the given integral can be determined.

To illustrate this take the integral

$$\int \sin^n x\,dx,$$

where n is a positive integer. Integrating by parts with $u = \sin^{n-1} x$, $dv = \sin x\, dx$, we get

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \cos^2 x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx$$

$$- (n-1) \int \sin^n x\, dx.$$

Transposing the last integral and dividing by n, we get

$$\int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx.$$

By successive application of this formula we can make $\int \sin^n x\, dx$ depend on $\int dx$ or $\int \sin x\, dx$ according as n is even or odd.

Example. $\int \sin^6 x\, dx.$

By the formula just proved

$$\int \sin^6 x\, dx = -\frac{\sin^5 x \cos x}{6} + \frac{5}{6} \int \sin^4 x\, dx$$

$$= -\frac{\sin^5 x \cos x}{6} + \frac{5}{6}\left[-\frac{\sin^3 x \cos x}{4} + \frac{3}{4} \int \sin^2 x\, dx\right]$$

$$= -\frac{\sin^5 x \cos x}{6} - \frac{5}{24} \sin^3 x \cos x - \frac{5}{16} \sin x \cos x + \frac{5}{16} x + C.$$

EXERCISES

1. $\int x \cos x\, dx.$
2. $\int x e^x\, dx.$
3. $\int x \ln x\, dx.$
4. $\int \sin^{-1} x\, dx.$

5. $\int \tan^{-1} x\, dx.$

6. $\int \ln\,(x + \sqrt{a^2 + x^2})\, dx.$

7. $\int x \sec^{-1} x\, dx.$

8. $\int x^3 e^{-x}\, dx.$

9. $\int (x - 1)^2 \sin x\, dx.$

10. $\int \sqrt{x^2 - a^2}\, dx.$

11. $\int \sqrt{a^2 + x^2}\, dx.$

12. $\int e^{2x} \sin 3\,x\, dx.$

13. $\int e^x \cos x\, dx.$

14. $\int \sin 2\,x \cos 3\,x\, dx.$

15. Prove the formula

$$\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n - 1} + \frac{n - 2}{n - 1}\int \sec^{n-2} x\, dx,$$

and use it to integrate $\int \sec^3 x\, dx.$

16. Prove the formula

$$\int (a^2 - x^2)^n\, dx = \frac{x(a^2 - x^2)^n}{2\,n + 1} + \frac{2\,na^2}{2\,n + 1}\int (a^2 - x^2)^{n-1}\, dx,$$

and use it to integrate

$$\int (a^2 - x^2)^{\frac{3}{2}}\, dx$$

CHAPTER XII

DEFINITE INTEGRALS

84. Summation. — Between $x = a$ and $x = b$ let $f(x)$ be a continuous function of x. Divide the interval between a and b into n equal parts

$$\Delta x = \frac{b - a}{n}$$

and let $x_1, x_2, \cdots, x_{n-1}$ be the points of division. Form the sum

$$f(a)\ \Delta x + f(x_1)\ \Delta x + f(x_2)\ \Delta x + \cdots + f(x_{n-1})\ \Delta x.$$

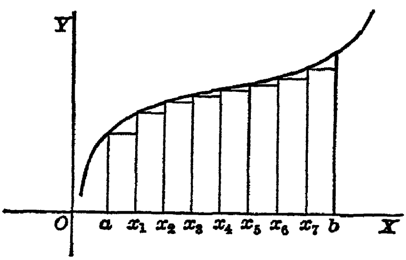

FIG. 84a.

This sum is represented by the notation

$$\sum\nolimits_a^b f(x)\ \Delta x.$$

Since $f(a)$, (fx_1), $f(x_2)$, etc., are the ordinates of the curve

$$y = f(x),$$

the terms $f(a)\,\Delta x$, $f(x_1)\,\Delta x$, $f(x_2)\,\Delta x$, etc., represent the areas of the rectangles in Fig. 84a and $\sum_a^b f(x)\,\Delta x$ is the sum of those rectangles.

Example 1. Find the value of $\sum_1^2 x^2\ \Delta x$ when $\Delta x = \frac{1}{4}$.

The interval between 1 and 2 is divided into parts of length $\Delta x = \frac{1}{4}$. The points of division are $1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$. Therefore

$$\begin{aligned}\sum\nolimits_1^2 x^2 \Delta x &= 1^2 \cdot \Delta x + (\tfrac{5}{4})^2\ \Delta x + (\tfrac{3}{2})^2\ \Delta x + (\tfrac{7}{4})^2\ \Delta x \\ &= \tfrac{63}{8}\ \Delta x = \tfrac{63}{8} \cdot \tfrac{1}{4} = 1.97.\end{aligned}$$

Example 2. Find approximately the area bounded by the x-axis, the curve $y = \sqrt{x}$, and the ordinates $x = 2$, $x = 4$.

From Fig. 84b it appears that a fairly good approximation will be obtained by dividing the interval between 2 and 4 into 10 parts each of length 0.2. The value of the area thus obtained is

$$\sum_2^4 \sqrt{x}\ \Delta x = (\sqrt{2} + \sqrt{2.2} + \sqrt{2.4} + \cdots + \sqrt{3.8})\,(0.2) = 3.39.$$

The area correct to two decimals (given by the method of Art. 87) is 3.45.

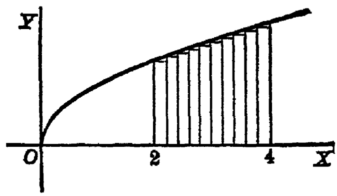

FIG. 84b.

85. Definite and Indefinite Integrals. — If we increase indefinitely the number of parts into which $b - a$ is divided, the intervals Δx approach zero and $\sum_a^b f(x)\,\Delta x$ usually approaches a limit. This limit is called the *definite* integral of $f(x)\,dx$ between $x = a$ and $x = b$. It is represented by the notation $\int_a^b f(x)\,dx$. That is

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_a^b f(x)\ \Delta x. \tag{85}$$

The number a is called the *lower limit*, b the *upper limit* of the integral.

In contradistinction to the definite integral (which has a definite value), the integral that we have previously used (which contains an undetermined constant) is called an *indefinite* integral. The connection between the two integrals will be shown in Art. 88.

86. Geometrical Representation. — If the curve $y = f(x)$ lies above the x-axis and $a < b$, as in Fig. 84a, $\int_a^b f(x)\,dx$ represents the limit approached by the sum of the inscribed rectangles and that limit is the area between $x = a$ and $x = b$ bounded by the curve and the x-axis.

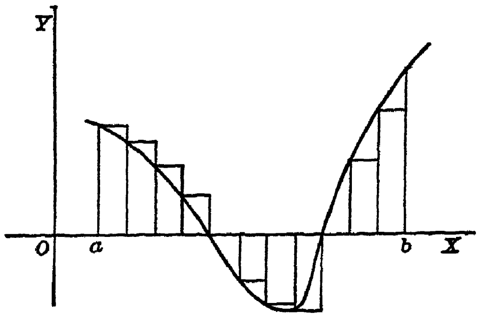

FIG. 86a.

At a point below the x-axis the ordinate $f(x)$ is negative and so the product $f(x)\Delta x$ is the negative of the area of the corresponding rectangle. Therefore (Fig. 86a)

$$\sum_a^b f(x)\,\Delta x = (\text{sum of rectangles above } OX) - (\text{sum of rectangles below } OX),$$

and in the limit

$$\int_a^b f(x)\,dx = (\text{area above } OX) - (\text{area below } OX).$$

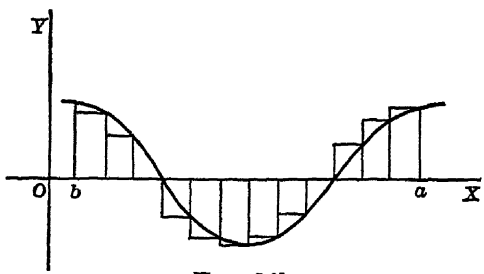

FIG. 86b.

If, however, $a > b$, as in Fig. 86b, x decreases as we pass from a to b, Δx is negative and instead of the above equation we have

$$\int_a^b f(x)\,dx = (\text{area below } OX) - (\text{area above } OX).$$

Example 1. Show graphically that $\int_0^{2\pi} \sin^3 x\,dx = 0.$

The curve $y = \sin^3 x$ is shown in Fig. 86c. Between $x = 0$ and $x = 2\pi$ the areas above and below the x-axis are equal. Hence

$$\int_0^{2\pi} \sin^3 x\,dx = A_1 - A_2 = 0.$$

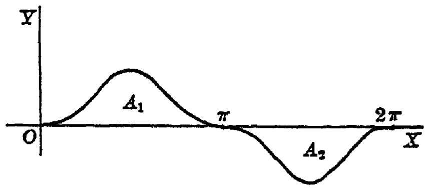

FIG. 86c.

Example 2. Show that

$$\int_{-1}^{1} e^{-x^2}\,dx = 2\int_0^1 e^{-x^2}\,dx.$$

The curve $y = e^{-x^2}$ is shown in Fig. 86d. It is symmetrical with respect to the y-axis. The area between $x = -1$ and

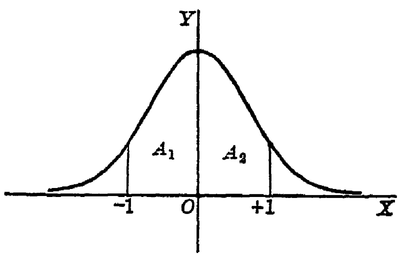

FIG 86d.

$x = 0$ is therefore equal to that between $x = 0$ and $x = 1$. Consequently

$$\int_{-1}^{1} e^{-x^2}\,dx = A_1 + A_2 = 2\,A_2 = 2\int_0^1 e^{-x^2}\,dx.$$

87. Derivative of Area. — The area A bounded by a curve

$$y = f(x),$$

a fixed ordinate $x = a$, and a movable ordinate MP, is a function of the abscissa x of the movable ordinate.

Let x change to $x + \Delta x$. The increment of area is

$$\Delta A = MPQN.$$

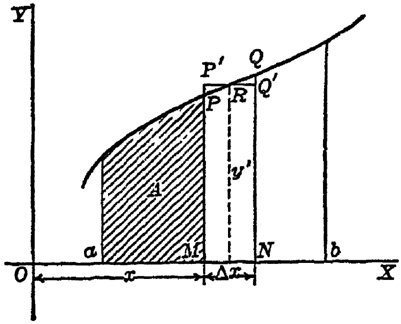

FIG. 87

Construct the rectangle $MP'Q'N$ equal in area to $MPQN$. If some of the points of the arc PQ are above $P'Q'$, others must be below to make $MPQN$ and $MP'Q'N$ equal. Hence $P'Q'$ intersects PQ at some point R. Let y' be the ordinate of R. Then y' is the altitude of $MP'Q'N$ and so

$$\Delta A = MPQN = MP'Q'N = y' \, \Delta x.$$

Consequently

$$\frac{\Delta A}{\Delta x} = y'.$$

When Δx approaches zero, if the curve is continuous, y' approaches y. Therefore in the limit

$$\frac{dA}{dx} = y = f(x). \tag{87a}$$

Let the indefinite integral of $f(x)\,dx$ be

$$\int f(x)\,dx = F(x) + C.$$

From equation (87a) we then have

$$A = \int f(x)\,dx = F(x) + C.$$

The area is zero when $x = a$. Consequently

$$0 = F(a) + C,$$

whence $C = -F(a)$ and

$$A = F(x) - F(a).$$

This is the area from $x = a$ to the ordinate MP with abscissa x. The area between $x = a$ and $x = b$ is then

$$A = F(b) - F(a). \tag{87b}$$

The difference $F(b) - F(a)$ is often represented by the notation $\Big[F(x)\Big]_a^b$, that is,

$$\Big[F(x)\Big]_a^b = F(b) - F(a). \tag{87c}$$

In the above discussion we have considered $f(x)$ as positive and $a < b$. If the curve extends below the x-axis as in Fig. 86a the same argument leads to equation (87b) if we consider area below the x-axis as negative. Finally, if $a > b$, as in Fig. 86b, the same equation is obtained if area below the x-axis is considered positive and above negative.

88. Relation of the Definite and Indefinite Integrals. — The definite integral $\int_a^b f(x)\,dx$ is equal to the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$, $x = b$. If

$$\int f(x)\,dx = F(x) + C,$$

by equation (87b) this area is $F(b) - F(a)$. We therefore conclude that

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a), \tag{88}$$

that is, *to find the value of the definite integral* $\int_a^b f(x)\,dx$, *substitute* $x = a$, *and* $x = b$ *in the indefinite integral* $\int f(x)\,dx$ *and subtract the former from the latter result.*

Example. Find the value of the integral

$$\int_0^1 \frac{dx}{1+x^2}.$$

The value required is

$$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}.$$

89. Properties of Definite Integrals. — A definite integral has the following simple properties:

I. $\int_a^b f(x)dx = -\int_b^a f(x)\,dx.$

II. $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$

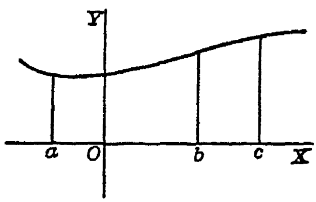

FIG. 89a.

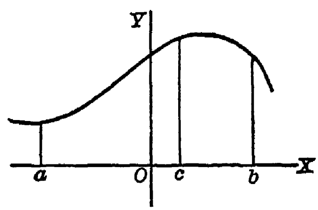

FIG 89b.

The first of these is due to the fact that if Δx is positive when x varies from a to b, it is negative when x varies from b to a. The two integrals thus represent the same area with different algebraic signs.

The second property expresses that the area from a to c is equal to the sum of the areas from a to b and b to c. This is the case not only when b is between a and c, as in Fig. 89a,

but also when b is beyond c, as in Fig. 89b. In the latter case $\int_b^c f(x)\,dx$ is negative and the sum

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

is equal to the difference of the two areas.

90. Infinite Limits. — It has been assumed that the limits a and b were finite. If the integral

$$\int_a^b f(x)\,dx$$

approaches a limit when b increases indefinitely, that limit is defined as the value of $\int_a^\infty f(x)\,dx$. That is,

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx. \tag{90}$$

If the indefinite integral

$$\int f(x)\,dx = F(x)$$

approaches a limit when x increases indefinitely,

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} [F(b) - F(a)] = F(\infty) - F(a).$$

The value is thus obtained by equation (88) just as if the limits were finite.

Example 1. $\displaystyle\int_0^\infty \frac{dx}{1+x^2}.$

The indefinite integral is

$$\int \frac{dx}{1+x^2} = \tan^{-1} x.$$

When x increases indefinitely, this approaches $\dfrac{\pi}{2}$. Hence

$$\int_0^\infty \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^\infty = \frac{\pi}{2}.$$

Example 2. $\int_0^\infty \cos x\, dx.$

The indefinite integral $\sin x$ does not approach a limit when x increases indefinitely. Hence

$$\int_0^\infty \cos x\, dx$$

has no definite value.

91. Infinite Values of the Function. — If the function $f(x)$ becomes infinite when $x = b$, $\int_a^b f(x)\, dx$ is defined as the limit

$$\int_a^b f(x)\, dx = \lim_{z \to b} \int_a^z f(x)\, dx,$$

z being between a and b.

Similarly, if $f(a)$ is infinite,

$$\int_a^b f(x)\, dx = \lim_{z \to a} \int_z^b f(x)\, dx,$$

z being between a and b.

If the function becomes infinite at a point c between a and b, $\int_a^b f(x)\, dx$ is defined by the equation

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx. \qquad (91)$$

Example 1. $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}.$

When $x = 0$, $\frac{1}{\sqrt[3]{x}}$ is infinite. We therefore divide the integral into two parts:

$$\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^1 \frac{dx}{\sqrt[3]{x}} = -\frac{3}{2} + \frac{3}{2} = 0.$$

Example 2. $\int_{-1}^{1}\frac{dx}{x^2}$.

If we use equation (88), we get

$$\int_{-1}^{1}\frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{1} = -2.$$

Since the integral is obviously positive, the result -2 is absurd. This is due to the fact that $\frac{1}{x^2}$ becomes infinite when $x = 0$. Resolving the integral into two parts, we get

$$\int_{-1}^{1}\frac{dx}{x^2} = \int_{-1}^{0}\frac{dx}{x^2} + \int_{0}^{1}\frac{dx}{x^2} = \infty + \infty = \infty.$$

92. Change of Variable. — If a change of variable is made in evaluating an integral, the limits can be replaced by the corresponding values of the new variable. To see this, suppose that when x is expressed in terms of t,

$$f(x)\,dx = \phi(t)\,dt.$$

Let

$$\int f(x)\,dx = F(x), \qquad \int \phi(t)\,dt = \Phi(t).$$

Then

$$d\,[F(x) - \Phi(t)] = f(x)\,dx - \phi(t)\,dt = 0.$$

Hence $F(x) - \Phi(t)$ is constant, or

$$F(x) = \Phi(t) + C.$$

If t_0, t_1 are values of t corresponding to x_0, x_1

$$F(x_1) = \Phi(t_1) + C, \qquad F(x_2) = \Phi(t_2) + C.$$

Hence

$$F(x_2) - F(x_1) = \Phi(t_2) - \Phi(t_1),$$

or

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{t_1}^{t_2}\phi(t)\,dt,$$

which was to be proved.

If more than one value of t corresponds to the same value of x, care should be taken to see that when t varies from t_0 to t_1, x varies from x_0 to x_1, and that for all intermediate values, $f(x)\,dx = \phi(t)\,dt$.

Example. $\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx.$

Substituting $x = a \sin\theta$, we find

$$\int \sqrt{a^2 - x^2}\,dx = a^2 \int \cos^2\theta\,d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right].$$

When $x = a$, $\sin\theta = 1$, and $\theta = \frac{\pi}{2}$. When $x = -a$, $\sin\theta = -1$ and $\theta = -\frac{\pi}{2}$. Therefore

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi a^2}{2}.$$

Since $\sin \frac{3}{2}\pi = -1$, it might seem that we could use $\frac{3}{2}\pi$ as the lower limit. We should then get

$$a^2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta = -\frac{\pi a^2}{2}.$$

This is not correct because in passing from $\frac{3}{2}\pi$ to $\frac{1}{2}\pi$, θ crosses the third and second quadrants. There $\cos\theta$ is negative and

$$\sqrt{a^2 - x^2}\,dx = (-a\cos\theta)\cos\theta\,d\theta,$$

and not $a^2\cos^2\theta\,d\theta$ as assumed above.

EXERCISES

Find the values of the following sums:

1. $\sum_0^3 x\,\Delta x$, $\quad \Delta x = 0.3$. **2.** $\sum_1^{10} \frac{\Delta x}{x}$, $\quad \Delta x = 1$.

3. $\sum_0^{\frac{\pi}{6}} \sin x\,\Delta x$, $\quad \Delta x = \frac{\pi}{36}$.

4. By summation with $\Delta x = 0.1$ find an approximate value of the area bounded by the x-axis, the ordinates $x = 0$, $x = 1$, and the curve $y = x^2$.

5. Find approximately the area bounded by the coördinate axes and a quadrant of the circle $x^2 + y^2 = 4$.

By representing the integrals as areas prove graphically the following equations:

6. $\displaystyle\int_0^{\pi} \sin 2x\, dx = 0.$

7. $\displaystyle\int_{-a}^{a} x\sqrt{a^2 - x^2}\, dx = 0.$

8. $\displaystyle\int_0^{\pi} \sin^5 x\, dx = 2\int_0^{\frac{\pi}{2}} \sin^5 x\, dx.$

9. $\displaystyle\int_0^{a} f(x)\, dx = \int_0^{a} f(a - x)\, dx.$

Find the values of the following definite integrals:

10. $\displaystyle\int_1^{2} (x^2 - 2x + 3)\, dx.$

11. $\displaystyle\int_0^{1} \frac{dx}{x + 1}.$

12. $\displaystyle\int_0^{\frac{\pi}{3}} \sin x\, dx.$

13. $\displaystyle\int_1^{2} \sqrt{x - 1}\, dx.$

14. $\displaystyle\int_1^{\infty} \frac{dx}{x^2}.$

15. $\displaystyle\int_0^{4} \frac{dx}{\sqrt{x}}.$

16. $\displaystyle\int_0^{\infty} e^{-x^2} x\, dx.$

Evaluate the following definite integrals by making the changes of variable indicated:

17. $\displaystyle\int_0^{4} \frac{dx}{\sqrt{x}(1 + x)}, \qquad x = z^2.$

18. $\displaystyle\int_1^{2} \frac{x\, dx}{\sqrt{x - 1}}, \qquad x - 1 = z^2.$

19. $\displaystyle\int_1^{\infty} \frac{dx}{x\sqrt{x^2 - 1}}, \qquad x = \sec\theta.$

20. $\displaystyle\int_0^{1} \frac{dx}{(1 + x^2)^{\frac{3}{2}}}, \qquad x = \tan \theta.$

CHAPTER XIII

SIMPLE AREAS AND VOLUMES

93. Area Bounded by a Plane Curve, Rectangular Coordinates. — If y is positive and $b > a$, the area bounded by the curve

$$y = f(x),$$

the x-axis, and two ordinates $x = a$, $x = b$ is the limit approached by the sum of rectangles $\sum_a^b y\,\Delta x$. That is,

$$A = \lim_{\Delta x \to 0} \sum_a^b y\,\Delta x = \int_a^b y\,dx = \int_a^b f(x)\,dx. \qquad (93a)$$

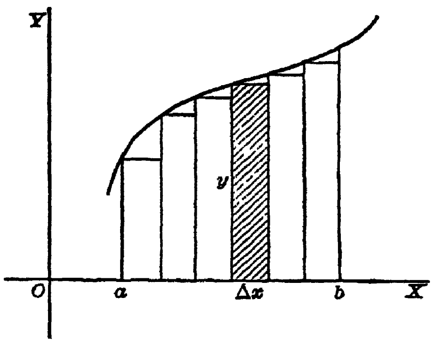

FIG. 93a.

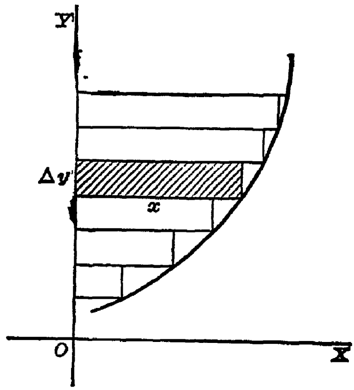

FIG. 93b.

Similarly, the area bounded by the curve

$$x = f(y)$$

the y-axis, and two abscissas $x = a$, $x = b$ is

$$A = \lim_{\Delta y \to 0} \sum_a^b x\,\Delta y = \int_a^b x\,dy = \int_a^b f(y)\,dy. \qquad (93b)$$

Areas of a more general form can be obtained by easy extensions of these formulas, the general method being to express the area as the limit of a sum of rectangles and determine that limit by integration.

Example 1. Find the area bounded by the curve $y=1+x^2$, the x-axis, and the ordinates $x = -1$, $x = 1$.

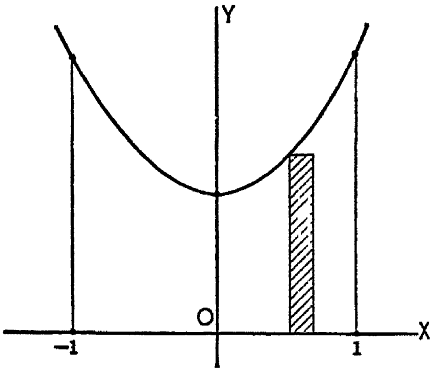

FIG. 93c.

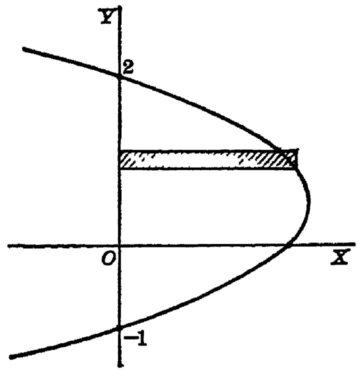

FIG. 93d.

The area (Fig. 93c) is

$$A = \int_{-1}^{1} y\,dx = \int_{-1}^{1} (1 + x^2)\,dx$$

$$= \left[x + \frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{3} - \left(-\frac{4}{3}\right) = 2\tfrac{2}{3}.$$

Example 2. Find the area bounded by the curve

$$x = 2 + y - y^2$$

and the y-axis.

The curve (Fig. 93d) crosses the y-axis at $y = -1$ and $y = 2$. The area required is therefore

$$A = \int_{-1}^{2} x\,dy = \int_{-1}^{2} (2 + y - y^2)\,dy = \left[2\,y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^{2} = 4\tfrac{1}{2}.$$

Example 3. Find the area bounded by the curves

$$y = 3\,x - x^2, \qquad y = x^2 - x.$$

By solving simultaneously we find that these curves (Fig. 93e) intersect at the origin and at the point Q where $x = 2, \quad y = 2$. The area required is bounded above by the arc OPQ of the curve $y = 3x - x^2$ and below by the

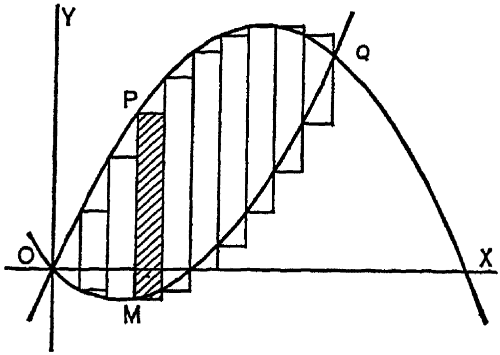

FIG. 93e.

arc OMQ of the curve $y = x^2 - x$. The area is the limit of a sum of rectangles with base Δx and altitude MP equal to the difference of the ordinates on the two curves. Hence

$$A = \int_0^2 MP \cdot dx = \int_0^2 [(3x - x^2) - (x^2 - x)]\, dx$$
$$= \int_0^2 (4x - 2x^2)\, dx = \tfrac{8}{3}.$$

Example 4. Find the area within the hypocycloid

$$x = a \sin^3 \phi, \qquad y = a \cos^3 \phi.$$

The area OAB in the first quadrant is

$$\int_0^a y\, dx,$$

where x and y have the values on the curve. To evaluate this integral we express x and y in terms of ϕ and change the limits as explained in Art. 92. When ϕ varies from 0 to $\frac{\pi}{2}$

the point (x, y) moves along the curve from A to B. Hence the area OAB is

$$\int y\,dx = \int_0^{\frac{\pi}{2}} a\cos^3\phi \cdot 3\,a\sin^2\phi\cos\phi\,d\phi$$

$$= 3\,a^2\int_0^{\frac{\pi}{2}} \cos^4\phi\sin^2\phi\,d\phi = \tfrac{3}{32}\,\pi\,a^2.$$

The entire area within the curve is

$$4 \cdot OAB = \tfrac{3}{8}\,\pi\,a^2.$$

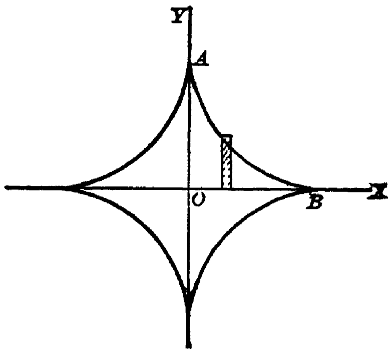

FIG. 93f.

EXERCISES

1. Find the area bounded by the line $y = 2\,x - 1$, the x-axis, and the ordinates $x = 1$, $x = 2$.

2. Find the area bounded by the x-axis and the curve

$$y = 2 - x - x^2.$$

3. Find the area bounded by the parabola

$$x = 2\,y - y^2$$

and the y-axis.

4. Find the area bounded by the curve

$$y^3 = x,$$

and the lines $y = -2$, $x = 8$.

5. Find the area bounded by the parabola

$$y = 2\,x - x^2$$

and the line $y = -x$.

6. Find the area bounded by the curve

$$y = 5 - x^2$$

and the line $y = x - 1$.

7. Find the area above the x-axis bounded by

$$y^2 = 4x, \qquad x + y = 3.$$

8. Find the area bounded by the curves

$$y^2 = 5x + 6, \qquad x^2 = y.$$

9. Find the area bounded by

$$xy = 4, \qquad x + y = 5.$$

10. Find the area bounded by

$$(x + y)^2 = 4x$$

and the line $y = -3$.

11. Find the area within the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

12. Find the area of the two parts into which the circle $x^2 + y^2 = 8$ is cut by the parabola $y^2 = 2x$.

13. Find the area within the ellipse

$$2x^2 - 2xy + y^2 = 4.$$

14. Find the area within the ellipse

$$x = 2\cos\phi, \qquad y = \sin\phi.$$

15. Find the area bounded by the hyperbola

$$x = a\sec\phi, \qquad y = a\tan\phi$$

and the line $x = 2a$.

16. Find the area bounded by the x-axis and one arch of the cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi).$$

17. Find the area of the sector bounded by the x-axis, the hyperbola

$$x = \frac{a}{2}(e^{\phi} + e^{-\phi}), \qquad y = \frac{a}{2}(e^{\phi} - e^{-\phi}),$$

and the line joining the origin to the point (x, y) on the curve.

94. Area Bounded by a Plane Curve, Polar Coördinates. — We shall determine the area of the sector POQ bounded by

two radii OP, OQ and the arc PQ of a given curve. Areas of a more general form can be expressed as sums or differences of areas of this kind.

Divide the angle POQ into equal parts $\Delta\theta$ and construct the circular sectors shown in Fig. 94a. One of these sectors ORS has the area

$$\tfrac{1}{2} OR^2 \cdot \Delta\theta = \tfrac{1}{2} r^2 \, \Delta\theta.$$

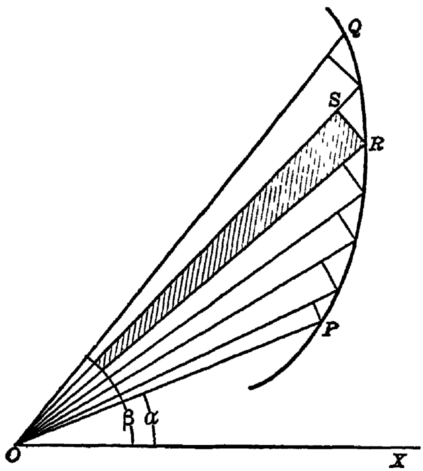

FIG. 94a.

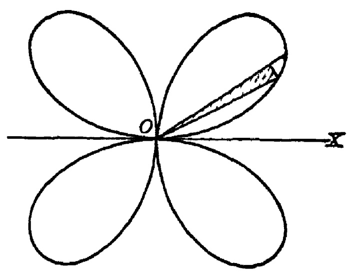

FIG 94b.

If α and β are the limiting values of θ, the sum of all the sectors is then

$$\sum_\alpha^\beta \tfrac{1}{2} r^2 \, \Delta\theta.$$

As $\Delta\theta$ approaches zero, this sum approaches the area A of the sector POQ. Therefore

$$A = \lim_{\Delta\theta \to 0} \sum_\alpha^\beta \tfrac{1}{2} r^2 \, \Delta\theta = \int_\alpha^\beta \tfrac{1}{2} r^2 \, d\theta. \tag{94}$$

In this equation r must be replaced by its value in terms of θ from the equation of the curve.

Example. Find the area of one loop of the curve $r = a \sin 2\,\theta$ (Fig. 94b).

A loop of the curve extends from $\theta = 0$ to $\theta = \frac{\pi}{2}$. Its area is

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta = \int_0^{\frac{\pi}{2}} \frac{a^2}{2} \sin^2 (2\,\theta) \, d\theta$$

$$= \frac{a^2}{4} \int_0^{\frac{\pi}{2}} (1 - \cos 4\,\theta) \, d\theta = \frac{\pi a^2}{8}.$$

EXERCISES

1. Find the area of the circle $r = a$.

2. Find the area of the circle $r = a \cos \theta$.

3. Find the area bounded by the coördinate axes and the line

$$r = a \sec \left(\theta - \frac{\pi}{3}\right).$$

4. Find the area of one loop of the curve

$$r^2 = a^2 \cos 2\,\theta.$$

5. Find the area enclosed by the curve $r = \cos \theta + 2$.

6. Find the area within the cardioid $r = a(1 - \cos \theta)$.

7. Find the area bounded by the y-axis and the parabola

$$r = a \sec^2 \frac{\theta}{2}.$$

8. Find the area bounded by the initial line and the second and third turns of the spiral $r = a\theta$.

9. Show that the area bounded by the spiral $r\theta = a$ and two radii is proportional to the difference of those radii.

10. Show that the area bounded by the spiral

$$r = ae^{k\theta}$$

and two radii is proportional to the difference of the squares of those radii.

11. Find the area below the initial line bounded by the circle

$$r = a \cos \theta + a \sin \theta.$$

12. Find the area common to the two circles

$$r = a \cos \theta, \qquad r = a \sin \theta.$$

13. Find the area above the x-axis bounded by the curve $r = a \tan \theta$ and the line $r = a \sec \theta$.

14. Find the area within the cardioid $r = a(1 + \cos \theta)$ and outside the circle $r = \frac{3}{2} a$.

15. By changing to polar coordinates find the area within a loop of the curve

$$(x^2 + y^2)^2 = 2\,a^2\,xy.$$

95. Volume of a Solid of Revolution. — To determine the volume generated by rotating the area $ABCD$ (Fig. 95a) about the x-axis, construct the series of rectangles shown in the diagram. When rotated about the x-axis one of these rectangles $PQRS$ generates a right cylinder with radius y and altitude Δx. The volume of this cylinder is

$$\pi y^2\,\Delta x.$$

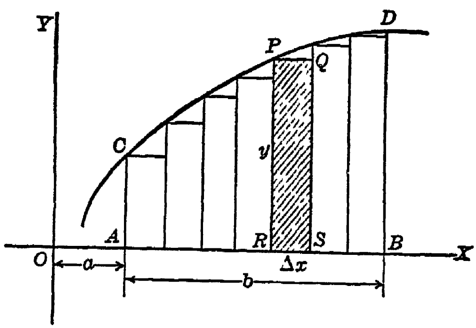

FIG. 95a.

If a and b are the limiting values of x, the sum of the cylinders is

$$\sum_a^b \pi y^2\,\Delta x.$$

The volume generated by the area is the limit of this sum

$$v = \lim_{\Delta x \to 0} \sum_a^b \pi y^2\,\Delta x = \int_a^b \pi y^2\,dx. \tag{95}$$

If the area does not reach the axis, as in Fig. 95b, let y_1 and y_2 be the distances from the axis to the bottom and top

of the rectangle $PQRS$. When revolved about the axis, it generates a hollow cylinder, or washer, of volume

$$\pi(y_2^2 - y_1^2)\,\Delta x.$$

The volume generated by the area is then

$$v = \lim_{\Delta x \to 0} \sum_a^b \pi(y_2^2 - y_1^2)\,\Delta x = \int_a^b \pi(y_2^2 - y_1^2)\,dx.$$

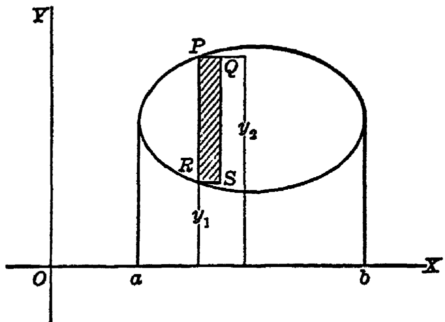

FIG 95b.

If the area is revolved about some other axis, y in these formulas must be replaced by the perpendicular from a point of the curve to the axis and x by the distance along the axis to that perpendicular.

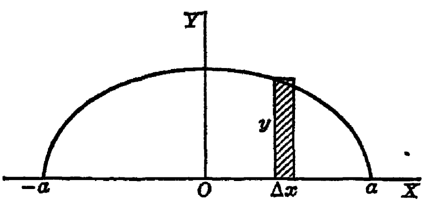

Fig. 95c.

Example 1. Find the volume generated by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis.

From the equation of the curve we get

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

The volume required is, therefore,

$$v = \int_{-a}^{a} \pi y^2\,dx = \frac{\pi b^2}{a^2}\int_{-a}^{a}(a^2 - x^2)\,dx = \tfrac{4}{3}\pi a b^2.$$

Example 2. A circle of radius a is revolved about an axis in its plane at the distance b (greater than a) from its center. Find the volume generated.

Revolve the circle, Fig. 95d, about the line CD. The rectangle MN generates a washer with radii

$$R_1 = b - x = b - \sqrt{a^2 - y^2},$$
$$R_2 = b + x = b + \sqrt{a^2 - y^2}.$$

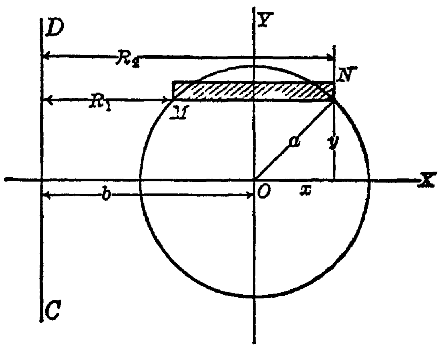

FIG. 95d.

The volume of the washer is

$$\pi\,(R_2^2 - R_1^2) = 4\,\pi b\,\sqrt{a^2 - y^2}\,\Delta y.$$

The volume required is then

$$v = \int_{-a}^{a} 4\,\pi b\,\sqrt{a^2 - y^2}\,dy = 2\,\pi^2 a^2 b.$$

Example 3. Find the volume generated by revolving the circle $r = a \sin\theta$ about the x-axis.

In this case

$$y = r\sin\theta = a\sin^2\theta,$$
$$x = r\cos\theta = a\cos\theta\sin\theta.$$

The volume required is

$$v = \int \pi y^2\,dx = \int_{\pi}^{0} \pi a^3 \sin^4\theta\,(\cos^2\theta - \sin^2\theta)\,d\theta = \frac{\pi^2 a^3}{4}.$$

The reason for using π as the lower limit and 0 as the upper is to make dx positive along the upper part ABC of the curve. As θ varies from π to 0, the point P describes the path $OABCO$. Along OA and CO dx is negative. The integral thus gives the volume generated by $MABCN$ minus that generated by OAM and OCN.

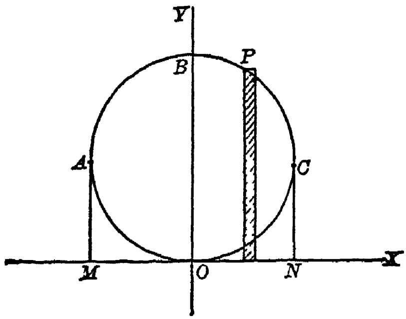

FIG. 95e.

EXERCISES

1. Find the volume of a sphere by integration.

2. Find the volume of a right cone by integration.

3. Find the volume generated by revolving about the x-axis the area bounded by the x-axis and the parabola $y = 2x - x^2$.

4. The area bounded by the parabola $y^2 = 4ax$ and the line $x = a$ is rotated about the line $x = 2a$. Find the volume generated.

5. The area bounded by the parabola $y^2 = 4ax$ and the line $x = a$ is revolved about the y-axis. Find the volume generated.

6. The area bounded by the parabola $y^2 = 4ax$ and the line $x = a$ is revolved about the line $y = -2a$. Find the volume generated.

7. Through a sphere of radius a a hole of radius b is bored. If the axis of the hole passes through the center of the sphere, find the volume left.

8. The area bounded by the hyperbola $xy = 4$ and the line $x + y = 5$ is revolved about the y-axis. Find the volume generated.

9. A loop of the curve

$$y^2 = x^2(1 - x^2)$$

is rotated about the y-axis. Find the volume generated.

10. The area within the ellipse

$$x = a \cos \phi, \qquad y = b \sin \phi$$

is rotated about the x-axis. Find the volume generated.

11. Find the volume generated by rotating about the x-axis the area between the x-axis and one arch of the cycloid

$$x = a(\phi - \sin \phi), \qquad y = a(1 - \cos \phi).$$

12. The area within the cardioid

$$r = a(1 + \cos \theta)$$

is revolved about the initial line. Find the volume generated.

96. Volume of a Solid with Given Area of Section. — Divide the solid (Fig. 96a) into slices of equal thickness Δh

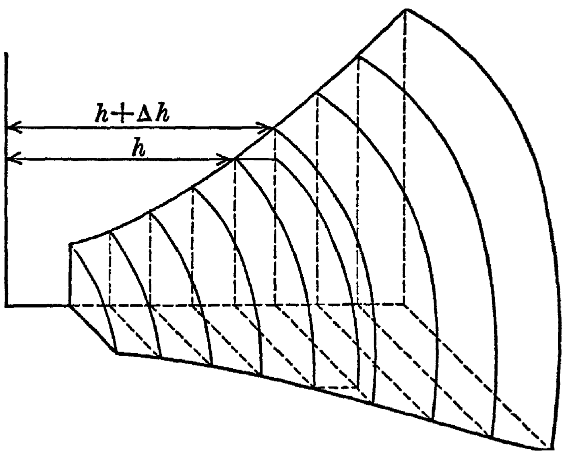

FIG. 96a.

by means of planes perpendicular to an axis along which a coördinate h is measured. Let A be the area of section in the plane where the coördinate is h. In the slice between the planes h, $h + \Delta h$ construct a plate with base A, altitude Δh, and lateral surface parallel to the axis of h. The volume of this plate is

$$A \cdot \Delta h.$$

If similar plates are inscribed in all the slices, the sum of their volumes is

$$\sum_a^b A\ \Delta h,$$

where a and b are the limiting values of h. When Δh approaches zero, this sum approaches the volume of the solid as limit. The volume is therefore

$$v = \lim_{\Delta h \to 0} \sum_a^b A\ \Delta h = \int_a^b A\ dh. \tag{96}$$

To determine the volume we must express A in terms of h and integrate. Particular care should be taken to choose the axis of h so that the area of section A shall be as simple a function of h as possible.

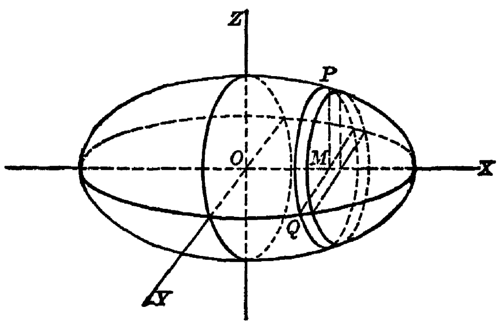

FIG. 96b.

Example 1. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The section perpendicular to the x-axis at the distance x from the center is an ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}.$$

The semi-axes of this ellipse are

$$MP = c\sqrt{1 - \frac{x^2}{a^2}}, \qquad MQ = b\sqrt{1 - \frac{x^2}{a^2}}.$$

By exercise 11, page 156, the area of this ellipse is

$$\pi \cdot MP \cdot MQ = \pi bc\left(1 - \frac{x^2}{a^2}\right).$$

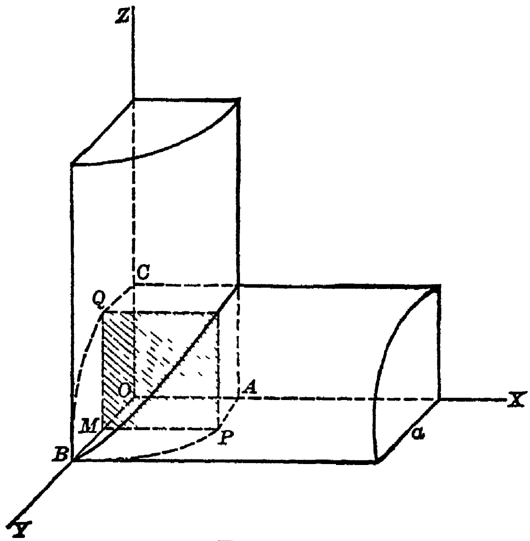

FIG 96c.

The volume of the ellipsoid is, therefore,

$$\int_{-a}^{a} \pi bc\left(1 - \frac{x^2}{a^2}\right)dx = \tfrac{4}{3}\,\pi abc.$$

Example 2. The axes of two equal right circular cylinders intersect at right angles. Find the common volume.

In Fig. 96c, the axes of the cylinders are OX and OZ and $OABC$ is $\frac{1}{8}$ of the common volume. The section of $OABC$ by a plane perpendicular to OY is a square of side

$$MP = MQ = \sqrt{a^2 - y^2}.$$

The area of section is therefore

$$MP \cdot MQ = a^2 - y^2$$

and the required volume is

$$v = 8 \int_0^a (a^2 - y^2)\, dy = \tfrac{16}{3} a^3.$$

EXERCISES

1. Find the volume of a pyramid by integration. Use the fact that a section parallel to the base has an area proportional to the square of its distance from the vertex.

2. Two circles have a diameter in common and lie in perpendicular planes. A square moves in such a way that its plane is perpendicular to the common diameter and its diagonals are chords of the circles. Find the volume generated.

3. A square of side a revolves about a line perpendicular to its plane while the point of intersection moves the distance h along the line. Find the volume generated.

4. The plane of a moving circle is perpendicular to that of an ellipse and the radius of the circle is an ordinate of the ellipse. Find the volume generated when the circle moves from one vertex of the ellipse to the other.

5. A wedge is cut from the base of a cylinder by a plane passing through a diameter of the base and inclined at the angle α to the base. Find the volume of the wedge.

6. A cylindrical bucket filled with oil is tipped until half the bottom is exposed. Find the amount of oil poured out.

7. The base of an oblique cylinder is a circle of radius a and the generators make the angle α with the base. A wedge is cut from the base of the cylinder by a plane which passes through the center of the base and is perpendicular to the generators. Find its volume.

8. The cylinder in the preceding problem is cut by a plane perpendicular to the generators and tangent to the base. Find the volume of the wedge cut off.

9. The axes of two equal right cylinders of radius a intersect at the angle α. Find the common volume.

10. Two oblique cylinders of equal altitude h have a circle of radius a as common upper base and their lower bases are tangent. Find the common volume.

11. The radius of a cylinder, starting from a position in the base, advances the distance $k\theta$ along the axis while rotating through the angle θ about it. If the total rotation is 180°, find the volume bounded by the resulting screw surface, the plane through the axis and final radius, the lateral surface of the cylinder, and its base.

CHAPTER XIV

OTHER GEOMETRICAL APPLICATIONS

97. Infinitesimals of Higher Order. — We have defined the definite integral as the limit

$$\lim_{\Delta x \to 0} \sum_a^b f(x)\ \Delta x.$$

In most applications of summation the quantity to be found appears as a limit of the form

$$\lim_{\Delta x \to 0} \sum_a^b F(x,\ \Delta x)$$

where $F(x,\ \Delta x)$ can be reduced to the form $f(x)\ \Delta x$ if we neglect infinitesimals of higher order than Δx. That such neglect should not change the limit is indicated by the following considerations.

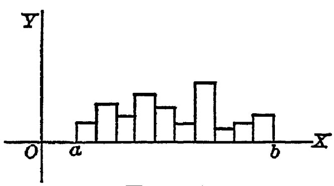

FIG. 97

Let ϵ be the number so chosen that

$$F(x,\ \Delta x) = f(x)\ \Delta x + \epsilon\ \Delta x.$$

There is a value of ϵ for each interval x, $x + \Delta x$. If the difference of $F(x,\ \Delta x)$ and $f(x)\ \Delta x$ is an infinitesimal of higher order than Δx, $\epsilon\ \Delta x$ is of higher order than Δx and so ϵ approaches zero with Δx (Art. 46). The difference

$$\sum_a^b F(x,\ \Delta x) - \sum_a^b f(x)\ \Delta x = \sum_a^b \epsilon\ \Delta x$$

is graphically represented by a sum of rectangles whose altitudes are the various values of ϵ. Since all these values

approach zero with Δx, the total area approaches zero and so

$$\lim_{\Delta x \to 0} \sum_a^b F(x, \Delta x) = \lim_{\Delta x \to 0} \sum_a^b f(x)\, \Delta x.$$

The quantity $f(x)\, \Delta x$ (usually written $f(x)\, dx$) is called the *element* of the integral.

For the above discussion to be strictly accurate it should be shown that there is a number m larger than any of the ϵ's which approaches zero. The small rectangles then all belong to a rectangle of altitude m, base $b-a$, and area $m(b-a)$ which approaches zero with Δx. In any actual application we can easily show that this is the case.

98. Length of a Curve, Rectangular Coördinates. — In the arc AB of a curve inscribe a series of chords. The length of one of these PQ (Fig. 98a) is

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

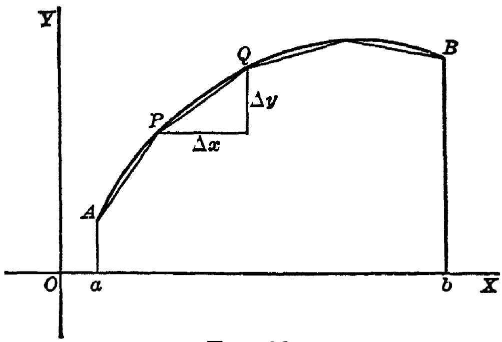

FIG. 98a.

and the sum of their lengths is

$$\sum_a^b \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

The length of the arc is defined as the limit approached by this sum when the number of chords is increased indefinitely, their separate lengths approaching zero.

If the slope of the curve is continuous, when Δx is sufficiently small,

$$\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

is approximately equal to

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

To see the accuracy of this approximation, let ϕ be the angle between the x-axis and the tangent at P and ϕ' the angle between the x-axis and the chord PQ (Fig. 98b). Then

$$\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} = \sec \phi', \qquad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sec \phi.$$

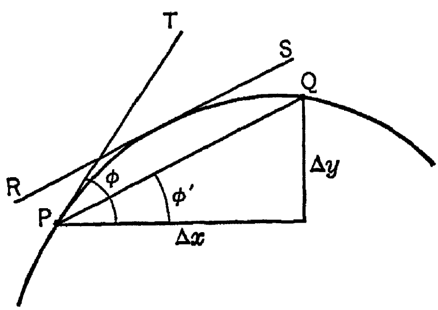

FIG. 98b.

On the arc PQ there is a tangent RS parallel to the chord PQ. Hence

$$\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} - \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sec \phi' - \sec \phi$$

where ϕ' and ϕ are the inclinations of two tangents on the arc PQ. Since ϕ is a continuous function, by making Δx sufficiently small we can make this difference as small as we please.

Hence in the above sum we can replace

$$\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

by

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

and so obtain for the length of the arc AB

$$s = \lim_{\Delta x \to 0} \sum_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, \Delta x = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

In applying this formula $\frac{dy}{dx}$ must be determined from the equation of the curve. The result can also be written

$$s = \int_A^B \sqrt{dx^2 + dy^2}. \tag{98}$$

In this formula y may be expressed in terms of x or x in terms of y or both in terms of a parameter. In any case the limits are the values at A and B of the variable that remains.

Example 1. Find the arc of the parabola $y^2 = 4\,x$ between the points (0, 0) and (1, 2).

In this case

$$dx = \tfrac{1}{2}\, y\, dy.$$

Hence

$$\sqrt{dx^2 + dy^2} = \sqrt{\tfrac{1}{4}\, y^2\, dy^2 + dy^2} = \tfrac{1}{2}\sqrt{y^2 + 4}\, dy,$$

$$s = \tfrac{1}{2}\int_0^2 \sqrt{y^2 + 4}\, dy = \sqrt{2} + \ln\left(1 + \sqrt{2}\right).$$

Example 2. Find the perimeter of the curve

$$x = a \cos^3 \phi, \qquad y = a \sin^3 \phi.$$

The differential of arc is

$$\begin{aligned}\sqrt{dx^2 + dy^2} &= \sqrt{9\, a^2 \cos^4 \phi \sin^2 \phi + 9\, a^2 \sin^4 \phi \cos^2 \phi}\, d\phi \\ &= 3\, a \cos \phi \sin \phi.\end{aligned}$$

One-fourth of the curve is described when ϕ varies from 0 to $\frac{\pi}{2}$. Hence the perimeter is

$$s = 4\int_0^{\frac{\pi}{2}} 3\,a\cos\phi\sin\phi\,d\phi = 6\,a.$$

99. Length of a Curve. Polar Coördinates. — The differential of arc of a curve is (Art. 62)

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dr^2 + r^2\,d\theta^2}.$$

Equation (98) is, therefore, equivalent to

$$s = \int_A^B \sqrt{dr^2 + r^2\,d\theta^2}. \tag{99}$$

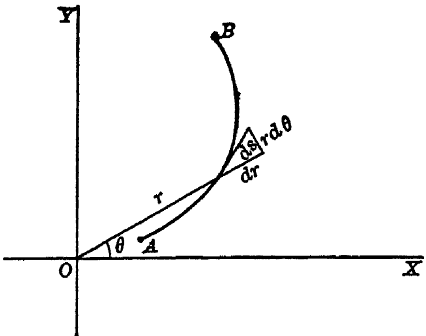

FIG. 99

In using this formula, r must be expressed in terms of θ or θ in terms of r from the equation of the curve. The limits are the values at A and B of the variable that remains.

Example. Find the length of one loop of the curve

$$r = a\cos^4\frac{\theta}{4}.$$

In this case

$$dr = -\,a\cos^3\frac{\theta}{4}\sin\frac{\theta}{4}\,d\theta$$

$$ds = \sqrt{dr^2 + r^2\,d\theta^2} = a\cos^3\frac{\theta}{4}\,d\theta.$$

One loop extends from $\theta = -2\,\pi$ to $\theta = 2\,\pi$. Hence

$$s = \int_{-2\pi}^{2\pi} a\cos^3\frac{\theta}{4}\,d\theta = \frac{16}{3}\,a.$$

EXERCISES

1. Find the length of the curve $y^2 = x^3$ between $(0, 0)$ and $(4, 8)$.

2. Find the arc of the curve $y^2 = (2x - 1)^3$ cut off by the line $x = 5$.

3. Find the length of the curve $x = \ln \sec y$ from $y = 0$ to $y = \frac{\pi}{3}$.

4. Find the length of the curve $x = \frac{1}{4} y^2 - \frac{1}{2} \ln y$ between $y = 1$ and $y = 2$.

5. Find the perimeter of the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

6. Find the length of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$$

between $x = -a$ and $x = a$.

7. Find the length of one arch of the cycloid

$$x = a(\phi - \sin \phi), \qquad y = a(1 - \cos \phi).$$

8. Find the length of the involute of the circle

$$x = a(\cos \theta + \theta \sin \theta), \qquad y = a(\sin \theta - \theta \cos \theta)$$

between $\theta = 0$ and $\theta = 2\pi$.

9. If $\theta \leq \pi$ and s is the arc of the cycloid

$$x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta)$$

between the origin and the point (x, y) on the curve, show that

$$s^2 = 8ay.$$

10. Find the circumference of the circle $r = a$ by integration.

11. Find the length of the curve $r = k\theta$ between $\theta = 0$ and $\theta = 2\pi$.

12. Find the circumference of the circle $r = 2a \cos \theta$.

13. Find the distance along the line $r = a \sec\left(\theta - \frac{\pi}{3}\right)$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

14. Show that the length of an arc of the spiral $r = ae^{b\theta}$ is proportional to the difference of the radii at its ends.

15. Find the arc of the parabola

$$r = a \sec^2 \tfrac{1}{2}\theta$$

cut off by the y-axis.

16. Find the perimeter of the cardioid

$$r = a(1 + \cos\theta).$$

17. Find the complete perimeter of the curve

$$r = a\sin^3\frac{\theta}{3}.$$

100. Area of a Surface of Revolution. — To find the area generated by revolving the arc AB about the x-axis join A and B by a broken line with vertices on the arc. Let x, y be the coördinates of P and $x + \Delta x$, $y + \Delta y$ those of Q. The chord PQ generates a frustum of a cone whose area is

$$\pi(2y + \Delta y)PQ = \pi(2y + \Delta y)\sqrt{\Delta x^2 + \Delta y^2}.$$

The area generated by the broken line is then

$$\sum_A^B \pi(2y + \Delta y)\sqrt{\Delta x^2 + \Delta y^2}.$$

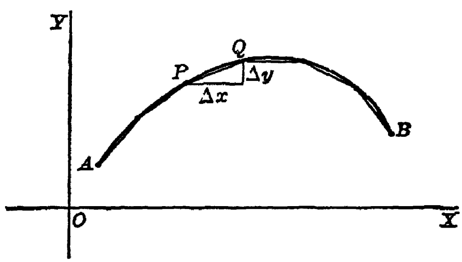

FIG. 100a.

The area S generated by the arc AB is the limit approached by this sum when Δx and Δy approach zero. Neglecting infinitesimals of higher order, $(2y + \Delta y)\sqrt{\Delta x^2 + \Delta y^2}$ can be replaced by $2y\sqrt{dx^2 + dy^2} = 2y\,ds$. Hence the area generated is

$$S = \int_A^B 2\pi y\,ds. \tag{100a}$$

In this formula y and ds must be calculated from the equation of the curve. The limits are the values at A and B of the variable in terms of which they are expressed.

Similarly, the area generated by revolving about the y-axis is

$$S = \int_A^B 2\pi x \, ds. \tag{100b}$$

Example 1. Find the area of the surface generated by revolving about the y-axis the part of the curve $y = 1 - x^2$ above the x-axis.

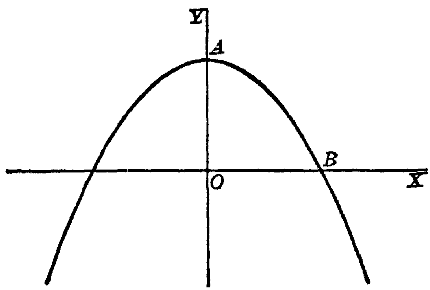

FIG. 100b.

In this case

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + 4x^2}\, dx.$$

The area required is generated by the part AB of the curve between $x = 0$ and $x = 1$. Hence

$$S = \int_A^B 2\pi x \, ds = \int_0^1 2\pi x \sqrt{1 + 4x^2}\, dx$$

$$= \frac{\pi}{6}(1 + 4x^2)^{\frac{3}{2}}\Big|_0^1 = \frac{\pi}{6}(5\sqrt{5} - 1).$$

Example 2. Find the area of the surface generated by rotating the cardioid

$$r = a(1 + \cos\theta)$$

about the initial line.

Using θ as the variable,

$$ds = \sqrt{dr^2 + r^2\,d\theta^2} = a\sqrt{2}\sqrt{1 + \cos\theta}\,d\theta,$$
$$y = r\sin\theta = a\,(1 + \cos\theta)\sin\theta.$$

The whole area is generated by the arc above the x-axis. Hence

$$S = \int 2\,\pi y\,ds = 2\,\pi a^2\sqrt{2}\int_0^{\pi}(1 + \cos\theta)^{\frac{3}{2}}\sin\theta\,d\theta$$
$$= \frac{4\,\pi a^2\sqrt{2}}{5}\Big[-(1 + \cos\theta)^{\frac{5}{2}}\Big]_0^{\pi} = \frac{32}{5}\pi a^2.$$

EXERCISES

1. Find the area of the surface of a sphere of radius a.

2. Find the area of the surface of a right circular cone of altitude h and radius of base a.

3. Find the area of the surface generated by revolving the arc of the curve

$$9\,y^2 = (2\,x - 1)^3$$

between $x = \frac{1}{2}$ and $x = 2$, about the y-axis

4. Find the area generated by rotating the loop of the curve

$$9\,x^2 = (2\,y - 1)\,(y - 2)^2$$

about the x-axis.

5. Find the area of the spheroid generated by rotating the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis.

6. Find the area of the surface generated by rotating the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

about the x-axis.

7. Find the area generated by rotating one arch of the cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi)$$

about the x-axis.

8. Find the area generated by rotating the ellipse

$$x = a\cos\phi, \qquad y = b\sin\phi$$

about the y-axis.

9. The arc of the hyperbola

$$x = a \sec \phi, \qquad y = b \tan \phi,$$

between $\phi = 0$ and $\phi = \frac{\pi}{4}$, is rotated about the y-axis. Find the area of the surface generated.

10. Find the area generated by rotating the circle $r = a \sin \theta$ about the initial line.

11. The arc of the curve $r = e^\theta$ between $\theta = 0$ and $\theta = \frac{\pi}{2}$ is revolved about the y-axis. Find the area of the surface generated.

12. The lemniscate

$$r^2 = 2\, a^2 \cos 2\, \theta$$

is revolved about the initial line. Find the area of the surface generated.

101. Unconventional Methods. — The methods that have been given for finding lengths, areas, and volumes are the ones most generally applicable. In particular cases other methods may give the results more easily. To solve a problem by integration, it is merely necessary to express the required quantity in any way as a limit of the form used in defining the definite integral.

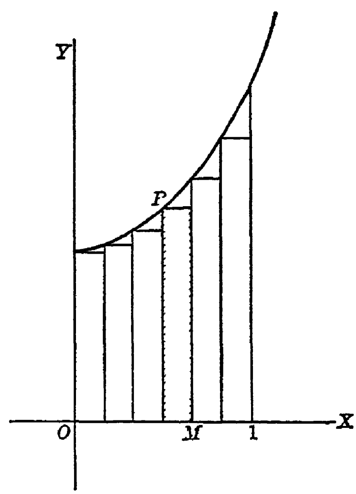

FIG. 101a.

Example 1. Find the volume generated by rotating about the y-axis the area bounded by the parabola $x^2 = y - 1$, the x-axis, and the ordinates $x = \pm 1$.

Resolve the area into slices by ordinates at distances Δx apart. When revolved about the y-axis, the rectangle PM between the ordinates x, $x + \Delta x$ generates a hollow cylinder whose volume is

$$\pi\, (x + \Delta x)^2 y - \pi x^2 y = 2\, \pi x y\, \Delta x + \pi y\, (\Delta x)^2.$$

Since $\pi y (\Delta x)^2$ is an infinitesimal of higher order than Δx, the required volume is

$$\lim_{\Delta\theta \to 0} \sum_0^1 2\,\pi x y\,\Delta x = \int_0^1 2\,\pi x\,(1 + x^2)\,dx = \tfrac{3}{2}\,\pi.$$

Example 2. When a string held taut is unwound from a fixed circle, its end describes a curve called the involute of the circle. Find the length of the part described when the first turn of the string is unwound.

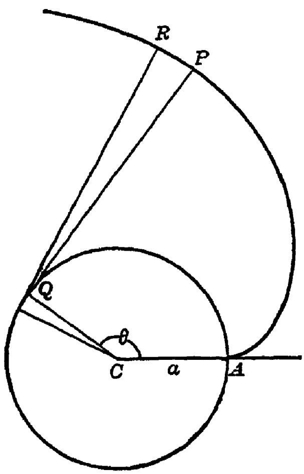

FIG. 101b.

Let the string begin to unwind at A. When the end reaches P the part unwound QP is equal to the arc AQ. Hence

$$QP = AQ = a\theta.$$

When P moves to R the arc PR is approximately the arc of a circle with center at Q and central angle $\Delta\theta$. Hence

$$PR = a\theta\,\Delta\theta$$

approximately. The length of the curve described when θ varies from 0 to $2\,\pi$ is then

$$s = \lim_{\Delta\theta \to 0} \sum_0^{2\pi} a\theta\,\Delta\theta = \int_0^{2\pi} a\theta\,d\theta = 2\,\pi^2 a.$$

Example 3. Find the area of the cylinder $x^2 + y^2 = ax$ within the sphere $x^2 + y^2 + z^2 = a^2$.

Fig. 101c shows one-fourth of the required area. Divide the circle OA into equal arcs Δs. The generators through the points of division cut the surface of the cylinder into strips. Neglecting infinitesimals of higher order, the area of the strip MPQ is $MP \cdot \Delta s$. If r, θ are the polar coördi-

nates of M, $r = a \cos \theta$ and

$$\Delta s = a\ \Delta\theta, \qquad MP = \sqrt{a^2 - r^2} = a \sin \theta.$$

The required area is therefore given by

$$\frac{S}{4} = \lim_{\Delta\theta \to 0} \sum_0^{\frac{\pi}{2}} a^2 \sin \theta\ \Delta\theta = \int_0^{\frac{\pi}{2}} a^2 \sin \theta\, d\theta.$$

Consequently

$$S = 4\, a^2 \int_0^{\frac{\pi}{2}} \sin \theta\, d\theta = 4\, a^2.$$

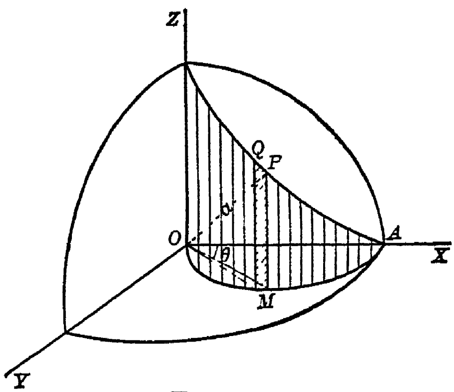

FIG. 101c.

EXERCISES

1. A hole of radius b is bored through a sphere of radius a. If the axis of the hole passes through the center of the sphere, find the volume cut from the sphere.

2. The area bounded by the hyperbola $x^2 - y^2 = a^2$ and the lines $y = \pm a$ is rotated about the x-axis. Find the volume generated.

3. The vertex of a cone of vertical angle $2\,\alpha$ is the center of a sphere of radius a. Find the volume common to the cone and sphere.

4. Find the area swept over by the string in Example 2, page 178.

5. A cycloid is generated by a point P on a circle of center C and radius a which rolls along a fixed straight line. Find the length of one arch of the cycloid by considering the circle at each instant as rotating about the point N where it touches the fixed line. Observe that the

amount the circle revolves is measured by the angle $\phi = NCP$ and not by the angle between NP and the fixed line.

6. Find the area between the fixed line and the cycloid in the preceding problem by finding the area swept over by the line NP. Observe that N and P are both moving.

7. Find the area of the surface cut from a right circular cylinder of radius a by a plane passing through a diameter of the base and inclined 45° to the base.

8. The axes of two right circular cylinders of radius a intersect at right angles (Fig. 96c). Find the area of the solid common to the two cylinders.

9. The angle between the axis of a cone and its generators is 45°. If the vertex of the cone is on the base of a cylinder of radius a and its axis is a generator of the cylinder, find the area of the cylindrical surface below the cone.

10. In Ex. 9 find the area of the conical surface within the cylinder.

11. In Ex. 9 find the volume of the cylinder below the cone.

12. The vertex of a cone is on the surface of a sphere of radius a and its axis is tangent to the sphere. If the vertical angle of the cone is 2α, find the area of its surface within the sphere.

CHAPTER XV

MECHANICAL AND PHYSICAL APPLICATIONS

102. Pressure. — The pressure of a liquid upon a horizontal area is equal to the weight of a vertical column of the liquid having the area as base and reaching to the surface. By the pressure at a point P in the liquid is meant the pressure upon a horizontal surface of unit area at that point. The volume of a column of unit section and height h is h. Hence the pressure at depth h is

$$p = wh, \tag{102a}$$

w being the weight of a cubic unit of the liquid.

To find the pressure upon one side of a vertical plane area (Fig. 102a), we make use of the fact that the pressure at a point is the same in all directions. The pressure upon the strip AB parallel to the surface is then approximately

$$p \, \Delta A,$$

p being the pressure at any point of the strip and ΔA its area. The reason for this not being exact is that the pressure at the top of the strip is a little less than at the bottom. This difference is, however, infinitesimal, and, since it multiplies ΔA, the error is an infinitesimal of higher order than ΔA. The total pressure is, therefore,

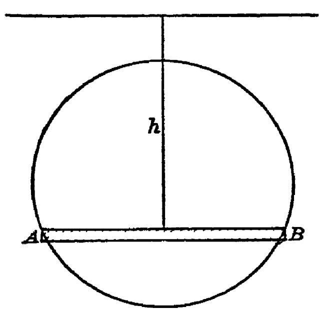

FIG 102a.

$$P = \lim_{\Delta A \to 0} \sum p \, \Delta A = \int p \, dA = w \int h \, dA. \tag{102b}$$

Before integration dA must be expressed in terms of h. The limits are the values of h at the top and bottom of the submerged area. In case of water the value of w is about 62.5 lbs. per cubic foot.

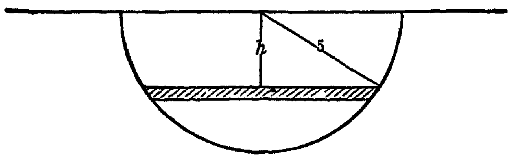

FIG 102b.

Example. Find the water pressure upon a semicircle of radius 5 ft., if its plane is vertical and its diameter in the surface of the water.

In this case the element of area is

$$dA = 2\sqrt{25 - h^2}\, dh.$$

Hence

$$P = w\int h\, dA = 2w\int_0^5 h\sqrt{25 - h^2}\, dh$$
$$= \tfrac{250}{3} w = \tfrac{250}{3}(62.5) = 5208.3 \text{ lbs.}$$

EXERCISES

1. Find the pressure sustained by a rectangular floodgate 10 ft. broad and 8 ft. deep, the upper edge being in the surface of the water.

2. Find the pressure on the lower half of the floodgate in the preceding problem.

3. Find the pressure on one side of a triangle of base b and altitude h, submerged so that its vertex is in the surface of the water, and its altitude vertical.

4. Find the pressure upon a triangle of base b and altitude h, submerged so that its base is in the surface of the liquid and its altitude vertical.

5. A square of edge 4 ft. is submerged with one diagonal vertical and its upper end in the surface of the water. Find the pressure on one side of the square.

6. Find the pressure on a semi-ellipse submerged with one axis (of length $2a$) in the surface and the other (of length $2b$) vertical.

7. Find the pressure on a parabolic segment of base $2b$ and altitude h, if the vertex is at the surface and the axis is vertical.

8. A vertical masonry dam is in the form of a trapezoid 300 ft. long at the surface, 200 ft. at the bottom, and 60 ft. high. What pressure must it withstand?

9. One end of a water main 3 ft. in diameter is closed by a vertical bulkhead. Find the pressure on the bulkhead if its center is 50 ft. below the surface of the water.

10. Find the pressure on one end of a cylindrical tank 4 ft. in diameter if its axis is horizontal and the tank is filled with water under a pressure of 2 lbs. per square foot at the top.

11. A rectangular tank is half filled with water and above this is oil. If oil is one-half as heavy as water show that the pressure on the sides is one-fourth greater than it would be if the tank were filled with oil.

12. A rectangle of altitude a and base b has the edge of length b in the surface of the water. Find the pressure on one side of the rectangle if its plane is inclined 30° to the vertical.

103. Moment. — If a force F acts in a plane perpendicular to an axis AB (Fig. 103a) the *moment* of the force about the axis is the product

$$M = Fl,$$

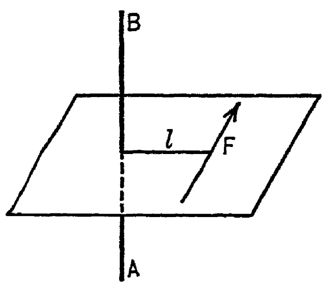

FIG. 103a.

where l is the lever arm, or perpendicular distance, from the axis to the line of the force. If a body is acted on by several forces, the total moment about a given axis is the sum of the moments of the separate forces, a moment being considered positive when the force tends to produce rotation in one direction about the axis and negative when it tends to produce rotation in the other direction.

The term moment is also applied to quantities other than forces, the moment in each case being the product of the magnitude of the quantity by its distance from an axis or plane.

In case of a plane area or length divide it into small parts such that the points of each differ only infinitesimally in

distance from an axis in the plane. Multiply each part by the distance of one of its points from the axis, the distance being considered positive for points on one side of the axis, negative for points on the other side. The limit approached by the sum of these products when the parts are taken smaller and smaller is called the moment of the length or area with respect to the axis.

Similarly, to find the moment of a length, area, volume, or mass in space with respect to a plane, we divide it into elements whose points differ only infinitesimally in distance from the plane and multiply each element by the distance of one of its points from the plane, these distances being considered positive for points on one side of the plane and negative for points on the other side. The moment with respect to the plane is the limit approached by the sum of these products when the elements are taken smaller and smaller.

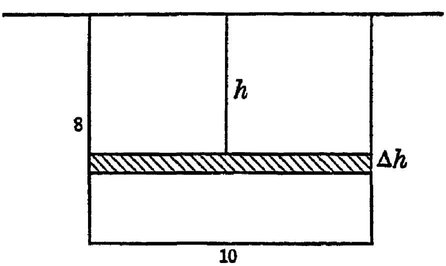

FIG. 103b.

Example 1. A rectangular floodgate is 10 ft. broad and 8 ft. deep. Find the moment of the water pressure about the base line if the water level is at the top of the gate.

Divide the rectangle into strips of width Δh by horizontal lines. Neglecting infinitesimals of higher order than Δh, the pressure on the strip (Fig. 103b) is

$$wh\,\Delta A = wh \cdot 10\,\Delta h.$$

The pressure being perpendicular to the area, its lever arm about the base line is

$$l = 8 - h.$$

Hence its moment about the base line is approximately

$$10\,wh\,(8 - h)\,\Delta h$$

and the total moment is

$$\lim_{\Delta h \to 0} \sum_0^8 10\, wh\,(8 - h)\ \Delta h = \int_0^8 10\, wh\,(8 - h)\, dh = 853\tfrac{1}{3}\, w.$$

Example 2. In the preceding example find the moment of the area about the base line.

The strip between the depths h, $h + \Delta h$ has the area

$$\Delta A = 10\ \Delta h.$$

All points in this strip are at distance approximately

$$l = 8 - h$$

from the base line. Except for infinitesimals of higher order, the moment of this strip is then

$$l\ \Delta A = 10\,(8 - h)\ \Delta h$$

and the total moment is

$$\lim_{\Delta h \to 0} \sum_0^8 10\,(8 - h)\ \Delta h = \int_0^8 10\,(8 - h)\ dh = 320.$$

104. Center of Gravity of a Length, Area, or Mass in a Plane. — The center of gravity is defined as the point at which the length, area, or mass could be concentrated without changing its moment with respect to any axis in the plane.

Let $C\,(\bar{x}, \bar{y})$ be the center of gravity of a mass. This may be the mass of a thin wire bent around a curve or the mass of a thin plate overlying an area. Cut the mass into pieces and let x, y be the coördinates of the center of gravity of the piece of mass Δm. The moment of this piece about the x-axis is $y\Delta m$ and the total moment of the whole body about the x-axis is

$$\lim_{\Delta m \to 0} \sum y \Delta m = \int y\, dm.$$

The mass of the body is

$$M = \int dm.$$

If this were concentrated at the center of gravity, its moment with respect to the x-axis would be $M\bar{y}$. By definition, we must then have

$$M\bar{y} = \int y\,dm,$$

whence

$$\bar{y} = \frac{\int y\,dm}{\int dm}. \tag{104a}$$

Similarly

$$\bar{x} = \frac{\int x\,dm}{\int dm}. \tag{104a}$$

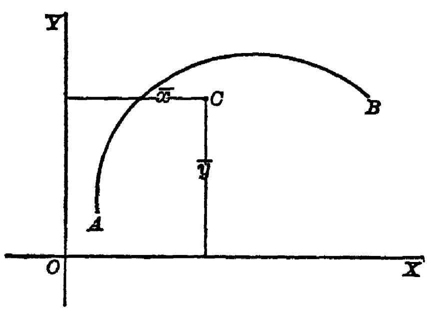

FIG. 104a.

In case of a thin wire (Fig. 104a)

$$dm = \rho\,ds$$

where ρ is the mass per unit length and ds is the element of arc. In particular, if ρ is constant, it may be canceled

from numerator and denominator, leaving

$$\bar{x} = \frac{\int x\,ds}{\int ds}, \qquad \bar{y} = \frac{\int y\,ds}{\int ds}. \tag{104b}$$

This is the center of gravity of the arc AB. The limits (not indicated) are the values at A and B of the variable in terms of which the integral is expressed.

In case of a thin plate

$$dm = \rho\,dA,$$

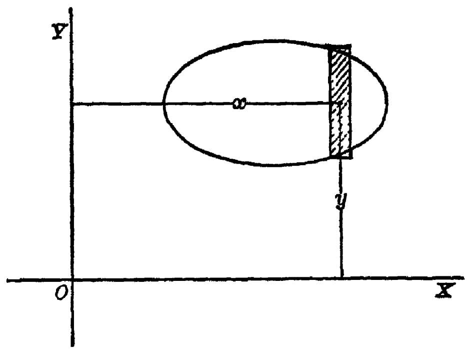

FIG. 104b.

where ρ is the mass per unit area and dA is the element of area. In particular, if ρ is constant, it may be canceled from the integrals, leaving

$$\bar{x} = \frac{\int x\,dA}{\int dA}, \qquad \bar{y} = \frac{\int y\,dA}{\int dA}. \tag{104c}$$

In these integrals x, y are the coördinates of the center of gravity of the element dm or dA. Before integration all quantities must be expressed in terms of a single variable and proper limits introduced. The element of area is

usually taken parallel to a coördinate axis, as in Figs. 104b and 104c, but any other form of element may be used if more convenient.

If the body is symmetrical with respect to an axis, its moment with respect to that axis is zero and so its center of gravity is on an axis of symmetry.

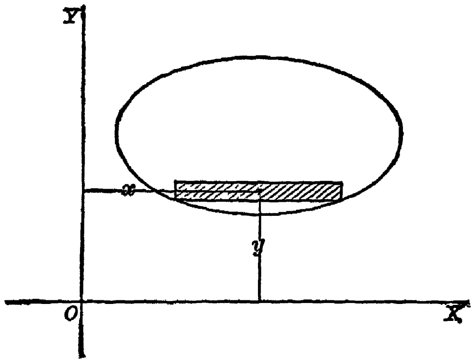

FIG. 104c.

Example 1. Find the center of gravity of a thin wire of uniform density and cross section bent into a quadrant of a circle of radius a.

In this case $x^2 + y^2 = a^2$,

$$ds = \sqrt{dx^2 + dy^2} = \frac{a}{y}\,dx,$$

and

$$\int y\,ds = \int_0^a y \cdot \frac{a}{y}\,dx = a^2,$$

$$\int ds = s = \frac{\pi}{2}a.$$

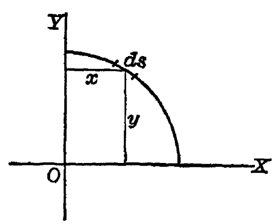

FIG. 104d.

Hence

$$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{2\,a}{\pi}.$$

It is evident from the symmetry of the figure that $\bar{x}$ has the same value.

Example 2. Find the center of gravity of the area of a semicircle of radius a.

From symmetry it is evident that the center of gravity is on the y-axis (Fig. 104e). Take the element of area parallel to OX. Then $dA = 2\,x\,dy$ and

$$\int y\,dA = \int 2\,xy\,dy = 2\int_0^a y\sqrt{a^2 - y^2}\,dy = \tfrac{2}{3}\,a^3.$$

The area is $A = \dfrac{\pi}{2}a^2$. Hence

$$\bar{y} = \frac{\int y\,dA}{A} = \frac{4\,a}{3\pi}, \qquad \bar{x} = 0.$$

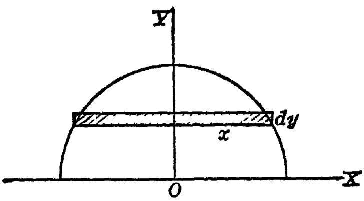

FIG. 104e.

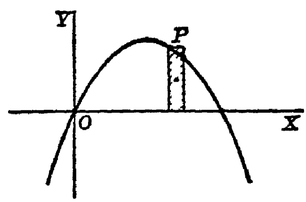

FIG. 104f.

Example 3. Find the center of gravity of the area bounded by the x-axis and the parabola $y = 2\,x - x^2$.

Take the element of area perpendicular to OX (Fig. 104f). If x, y are the coördinates of the top of the strip, its center of gravity is $(x, \frac{1}{2}\,y)$ and its moment with respect to the x-axis is

$$\frac{y}{2}dA = \tfrac{1}{2}\,y^2\,dx.$$

The moment of the whole area about the x-axis is then

$$\int \tfrac{1}{2}\,y^2\,dx = \int_0^2 \tfrac{1}{2}\,(2\,x - x^2)^2\,dx = \tfrac{8}{15}.$$

The area is

$$A = \int y\,dx = \int_0^2 (2x - x^2)\,dx = \tfrac{4}{3}.$$

Hence

$$\bar{y} = \frac{\int \frac{1}{2} y\,dA}{\int dA} = \frac{2}{5}.$$

Similarly

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^2 (2x^2 - x^3)\,dx}{A} = 1$$

as we might have anticipated since the figure is symmetrical with respect to the line $x = 1$.

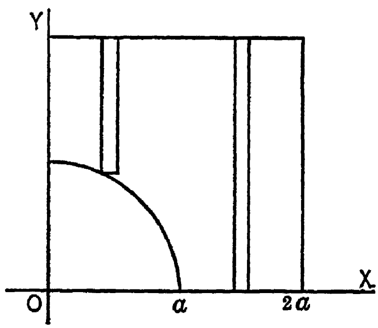

FIG. 104g.

Example 4. A thin plate has the form of a square of side $2a$ with a quadrant of a circle of radius a cut from one corner. Find its center of gravity.

Cut the plate into strips by lines $x =$ const. (Fig. 104g). The height of the strip depends on whether x is less than or greater than a. Between $x = 0$ and $x = a$ the height is

$$h = 2a - \sqrt{a^2 - x^2}.$$

Between $x = a$ and $x = 2a$ it is

$$h = 2a.$$

Hence

$$\int x\,dm = \int x\rho h\,dx = \int_0^a \rho x\left(2a - \sqrt{a^2 - x^2}\right)dx$$
$$+ \int_a^{2a} \rho x \cdot 2a\,dx = \rho\left(a^3 - \frac{a^3}{3}\right) + \rho\,(3a^3) = \frac{11}{3}\rho a^3.$$

The area (being the difference of a square and a quadrant of a circle) is

$$A = 4\,a^2 - \frac{\pi}{4}a^2.$$

Hence

$$\bar{x} = \frac{\int x\,dm}{M} = \frac{\frac{11}{3}\rho a^3}{\rho(4\,a^2 - \frac{\pi}{4}a^2)} = \frac{44}{3\,(16-\pi)}a.$$

From symmetry it is evident that $\bar{y}$ has the same value.

Example 5. Find the center of gravity of the area within the cardioid $r = a\,(1 + \cos\theta)$.

Use the polar element of area

$$dA = \tfrac{1}{2}\,r^2\,d\theta.$$

Such an element is approximately a triangle with center of gravity two-thirds of the way from the vertex to the base.

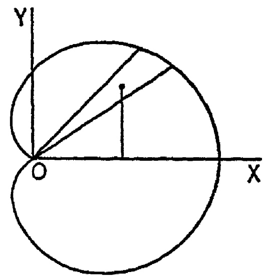

FIG 104h.

The coördinates of this point are

$$x = \tfrac{2}{3}\,r\cos\theta, \qquad y = \tfrac{2}{3}\,r\sin\theta.$$

Hence

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^{2\pi}\frac{1}{3}a^3(1+\cos\theta)^3\cos\theta\,d\theta}{\int_0^{2\pi}\frac{1}{2}a^2(1+\cos\theta)^2\,d\theta} = \frac{5}{6}a.$$

By symmetry it is clear that $\bar{y} = 0$.

105. Center of Gravity of a Length, Area, Volume, or Mass in Space. — The center of gravity is defined as the point at which the mass, area, length, or volume can be concentrated without changing its moment with respect to any plane.

Thus to find the center of gravity of a solid mass (Fig. 105a) cut it into slices of mass Δm. If (x, y, z) is the center of gravity of the slice, its moment with respect to the xy-plane is $z\ \Delta m$ and the moment of the whole mass is

$$\lim_{\Delta m \to 0} \sum z\ \Delta m = \int z\, dm.$$

If the whole mass M were concentrated at its center of gravity $(\bar{x}, \bar{y}, \bar{z})$, the moment with respect to the xy-plane would be $\bar{z}M$. Hence

$$\bar{z}M = \int z\, dm,$$

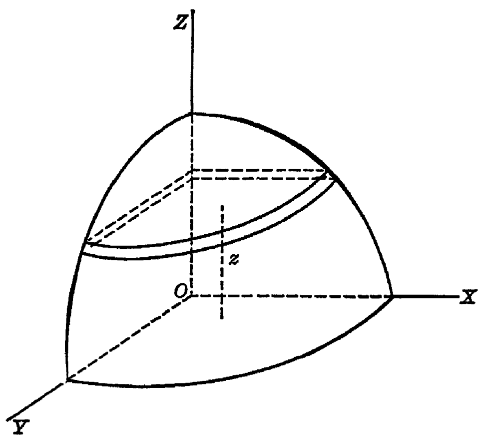

FIG 105a.

or

$$\bar{z} = \frac{\int z\, dm}{M}. \tag{105}$$

Similarly,

$$\bar{x} = \frac{\int x\, dm}{M}, \qquad \bar{y} = \frac{\int y\, dm}{M}. \tag{105}$$

The mass of a unit volume is called the density. If then dv is the volume of the element dm and ρ its density,

$$dm = \rho\, dv.$$

To find the center of gravity of a length, area, or volume it is merely necessary to replace M in these formulas by s, S, or v.

Example 1. Find the center of gravity of the volume of an octant of a sphere of radius a.

The volume of the slice (Fig. 105a) is

$$dv = \tfrac{1}{4}\,\pi x^2\, dz = \tfrac{1}{4}\,\pi\,(a^2 - z^2)\, dz.$$

Hence

$$\int z\, dv = \int_0^a \frac{1}{4}\pi\,(a^2 - z^2)\, z\, dz = \frac{\pi}{16}a^4.$$

The volume of an octant of a sphere is $\frac{1}{6}\,\pi a^3$. Hence

$$\bar{z} = \frac{\int z\, dv}{v} = \frac{\frac{\pi}{16}a^4}{\frac{\pi}{6}\,a^3} = \frac{3}{8}a.$$

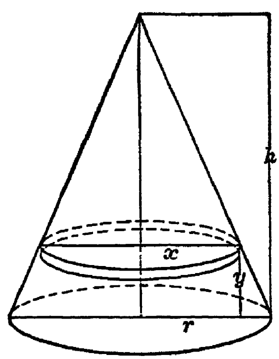

FIG 105b.

From symmetry it is evident that $\bar{x}$ and $\bar{y}$ have the same value.

Example 2. Find the center of gravity of a right circular cone whose density is proportional to the distance from its base.

Cut the cone into slices parallel to the base. Let y be the distance of a slice from the base. Except for infinitesimals of higher order its volume is $\pi x^2\, dy$, and its density is ky where k is constant. Hence its mass is

$$\Delta m = k\pi x^2 y\, dy.$$

By similar triangles $x = \frac{r}{h}(h - y)$. Hence

$$\int y\,dm = \int_0^h \frac{k\pi r^2}{h^2}(h - y)^2 y^2\,dy = \frac{k\pi r^2 h^3}{30}$$

$$M = \int dm = \int_0^h \frac{k\pi r^2}{h^2}(h - y)^2 y\,dy = \frac{k\pi r^2 h^2}{12}.$$

Therefore, finally,

$$\bar{y} = \frac{\int y\,dm}{M} = \frac{2}{5}h.$$

EXERCISES

1. The wind produces a uniform pressure of p lbs. per square foot on a door b ft. wide and h ft. high. Find the moment tending to turn the door on its hinges.

2. Find the moment of the pressure on a vertical floodgate of width b and height h about a horizontal line through its center when the water level is at the top of the gate.

3. Find the moment about the base line of the pressure on the dam in Ex. 8, page 183.

4. Weights of 1, 2 and 3 pounds are placed at the points (0, 0), (2, 1) and (4, 3). Find their center of gravity.

5. A cube of side 1 ft. and a cube of side 2 ft. are placed with their centers 3 ft. apart. If they are made of the same material, find their center of gravity. Make direct use of the fact that each cube can be considered as concentrated at its center.

6. A uniform wire is bent into a semicircle of radius a. Find its center of gravity.

7. Find the center of gravity of the arc of the curve

$$9y^2 = (2x - 1)^3$$

cut off by the line $x = 5$.

8. A thin triangular plate of constant density and thickness has a base b and altitude h Find the distance of its center of gravity from the base.

9. Find the center of gravity of a uniform plate having the form of a quadrant of a circle of radius a.

10. Find the center of gravity of the area cut from the parabola $y^2 = ax$ by the line $x = a$.

11. Find the center of gravity of the area bounded by the x-axis and the parabola $y = 1 - x^2$.

12. Find the center of gravity of the area bounded by the parabolas

$$y^2 = ax, \qquad x^2 = ay.$$

13. Find the center of gravity of the area above the x-axis bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

14. The two arms of a steel square are respectively 2 inches and $1\frac{1}{2}$ inches wide. If the outer edges are 24 inches and 12 inches long, find its center of gravity.

15. From a semicircle of radius b, a semicircle of radius a, with the same center, is cut. Find the center of gravity of the area left.

16. A thin plate has the form of a sector of a circle of radius a and central angle 2α. Find its center of gravity.

17. Find the center of gravity of a thin plate covering the area between the x-axis and one arch of the cycloid

$$x = a(\phi - \sin\phi), \qquad y = a(1 - \cos\phi).$$

18. Find the center of gravity of a uniform wire bent in the form of the arch of the cycloid in the preceding problem

19. Find the center of gravity of a hemispherical solid of constant density.

20. Find the center of gravity of a right circular cone of constant density.

21. From a cylinder of radius a and altitude h a cone of the same base and altitude is cut. Find the center of gravity of the remaining volume.

22. The area bounded by the parabola $y^2 = 4ax$ and the line $x = 2a$ is rotated about the x-axis. Find the center of gravity of the resulting volume.

23. The area above the x-axis bounded by the parabola $y^2 = 4ax$ and the line $x = 2a$ is revolved about the y-axis. Find the center of gravity of the resulting volume.

24. A hemisphere of radius a and a right circular cone have the same base and altitude. Find the center of gravity of the volume between the two surfaces

25. A hemispherical shell of constant density has an inner radius a and an outer radius b. Find its center of gravity.

26. Find the center of gravity of a thin shell covering the curved surface of a hemisphere of radius a.

27. Find the center of gravity of a thin shell covering the curved surface of a right circular cone of altitude h.

28. The base of a pyramid is a square of side a and the vertex is on the perpendicular to the base at one corner. Find the distance of the center of gravity of the pyramid from one of the side planes through that corner.

29. A wedge is cut from a right circular cylinder of radius a by a plane tangent to the base and making an angle α with the base. Find the distance of the center of gravity of the wedge from the base.

106. Moment of Inertia. — The moment of inertia of a particle about an axis is the product of its mass and the square of its distance from the axis.

To find the moment of inertia of a continuous mass, we divide it into parts such that the points of each differ only infinitesimally in distance from the axis. Let Δm be such a part and R the distance of one of its points from the axis. Except for infinitesimals of higher order, the moment of inertia of Δm about the axis is $R^2\,\Delta m$. The moment of inertia of the entire mass is therefore

$$I = \lim_{\Delta m \to 0} \sum R^2 \Delta m = \int R^2\, dm. \tag{106}$$

By the moment of inertia of a length, area, or volume, we mean the value obtained by using the differential of length, area, or volume in place of dm in equation (106).

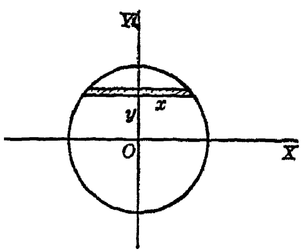

FIG. 106a.

Example 1. Find the moment of inertia of the area of a circle about a diameter of the circle.

Let the radius be a and let the x-axis be the diameter about which the moment of inertia is taken. Divide the area into strips by lines parallel to the x-axis. Neglecting infinitesimals of higher order, the area of such a strip is $2\,x\,\Delta y$ and its moment of inertia $2\,xy^2\,\Delta y$. The moment of inertia of the entire area is therefore

$$I = \int 2\,xy^2\,dy = 2\int_{-a}^{a} \sqrt{a^2 - y^2}\,y^2\,dy = \frac{\pi a^4}{4}.$$

Example 2. Find the moment of inertia of a right circular cone of constant density about its axis.

Let ρ be the density, h the altitude, and a the radius of the base of the cone. Divide it into hollow cylindrical slices by means of cylindrical surfaces having the same axis as the cone. By similar triangles the altitude y of the cylindrical surface of radius r is

$$y = \frac{h}{a}(a - r).$$

Neglecting infinitesimals of higher order, the volume between the cylinders of radii r and $r + \Delta r$ is then

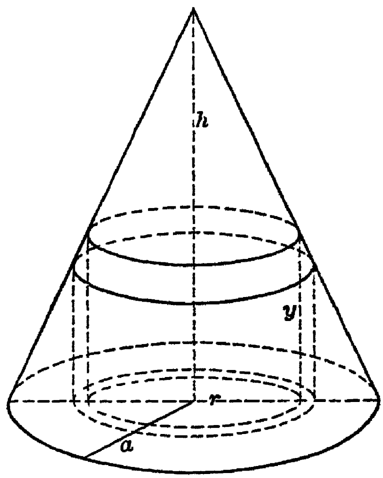

FIG. 106b.

$$\Delta v = 2\,\pi r y\,\Delta r = \frac{2\,\pi h}{a}\,r\,(a - r)\,dr.$$

The moment of inertia is therefore

$$I = \int r^2\,dm = \int r^2 \rho\,dv = \frac{2\,\pi h\rho}{a}\int_0^a r^3\,(a - r)\,dr = \frac{\pi\rho h a^4}{10}.$$

The mass of the cone is

$$M = \rho v = \tfrac{1}{3}\,\pi\rho a^2 h.$$

Hence

$$I = \tfrac{3}{10}\,Ma^2.$$

EXERCISES

1. The sides of a rectangle are a and b. Find the moment of inertia of its area about the side of length a.

2. Find the moment of inertia of a triangle of base b and altitude h about the axis through its vertex parallel to its base.

3. Find the moment of inertia of a triangle of base b and altitude h about its base.

4. Find the moment of inertia about the x-axis of the area bounded by the x-axis and the curve $y = 4 - x^2$.

5. Find the moment of inertia about the y-axis of the area bounded by the parabola $y^2 = 4\,ax$ and the line $x = a$.

6. A thin circular plate has a radius a and mass M. Find its moment of inertia about a diameter.

7. Find the moment of inertia of a uniform wire of mass M and length l about an axis perpendicular to the wire at one end.

8. A uniform wire of mass M is bent into a circle of radius a. Find its moment of inertia about a diameter of the circle.

9. Find the moment of inertia of the area of a circle about the axis perpendicular to its plane at its center. (Divide the area into rings with centers at the center of the circle.)

10. Show that the moment of inertia of a plane area about the axis perpendicular to its plane at the origin is equal to the sum of its moments of inertia with respect to the coördinate axis. Use this to find the moment of inertia of a thin square plate of mass M and side a about the axis perpendicular to its plane at its center.

11. By the method of the preceding problem find the moment of inertia of the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the axis perpendicular to its plane at its center.

12. Find the moment of inertia of a cylinder of mass M and radius a about its axis.

13. The rim of a flywheel has inner and outer radii r_1, r_2 and its mass is M. Find its moment of inertia about its axis.

14. From a right circular cylinder a right cone of the same radius and altitude is cut. If the radius is a and the altitude is h, find the moment of inertia of the remaining volume about the axis of the cylinder.

15. A volume is generated by rotating about the x-axis the area bounded by the parabola $y^2 = 4\,ax$ and the line $x = 2\,a$. Find its moment of inertia about the x-axis.

16. Find the moment of inertia of a spherical ball of radius a and mass M about a diameter.

17. A torus is generated by revolving a circle of radius a about an axis in its plane at distance b (greater than a) from its center. Find the moment of inertia of its volume about its axis.

18. A thin conical shell of constant thickness has an altitude h, radius of base a, and mass M. Find its moment of inertia about its axis.

19. A thin spherical shell has a radius a and mass M. Find its moment of inertia about a diameter.

20. The kinetic energy of a moving mass is

$$\int \tfrac{1}{2} v^2\, dm,$$

where v is the velocity of the element of mass dm. Find the kinetic energy of a homogeneous cylinder of radius a and mass M rotating with angular velocity ω about its axis.

21. Find the kinetic energy of a homogeneous sphere of radius a and mass M rotating with angular velocity ω about a diameter.

107. Work Done by a Force. — Let a force be applied to a body at a fixed point. When the body moves work is done by the force. If the force is constant, the work is defined as the product of the force and the distance the point of application moves in the direction of the force. That is,

$$W = Fs, \tag{107a}$$

where W is the work, F the force, and s the distance moved in the direction of the force.

If the direction of motion does not coincide with that of the force, the work done is the product of the force and the projection of the displacement on the force. Thus when the body moves from A to B (Fig. 107a) the work done by the force F is

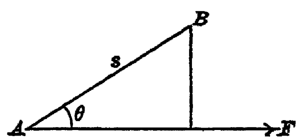

FIG 107a

$$W = Fs \cos \theta. \tag{107b}$$

If the force is variable, we divide the path into parts Δs. In moving the distance Δs, the force is nearly constant and so the work done is approximately $F \cos \theta\, \Delta s$. As the intervals Δs are taken shorter and shorter, this approximation becomes more and more accurate. The exact work is then the limit

$$W = \lim_{\Delta s \to 0} \sum F \cos \theta\, \Delta s = \int F \cos \theta\, ds. \tag{107c}$$

To determine the value of W, we express $F \cos\theta$ and ds in terms of a single variable. The limits of integration are the values of this variable at the two ends of the path. If the displacement is in the direction of the force, $\theta = 0$, $\cos\theta = 1$ and

$$W = \int F\, ds. \tag{107d}$$

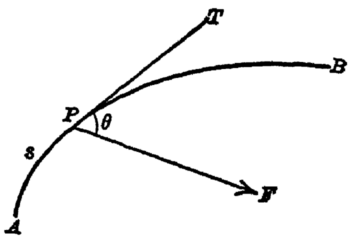

FIG. 107b.

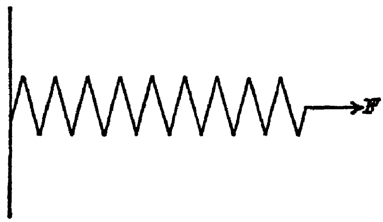

FIG. 107c

Example 1. The amount a helical spring is stretched is proportional to the force applied. If a force of 100 lbs. is required to stretch the spring 1 inch, find the work done in stretching it 4 inches.

Let s be the number of inches the spring is stretched. The force then is

$$F = ks,$$

k being constant. When $s = 1$, $F = 100$ lbs. Hence $k = 100$ and

$$F = 100\, s.$$

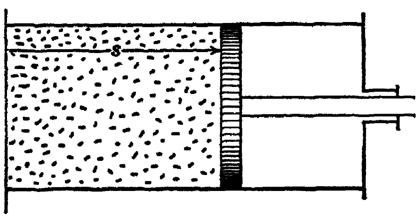

FIG. 107d.

The work done in stretching the spring 4 inches is

$$\int_0^4 F\, ds = \int_0^4 100\, s\, ds = 800 \text{ inch pounds} = 66\tfrac{2}{3} \text{ foot pounds.}$$

Example 2. A gas is confined in a cylinder with a movable piston. Assuming Boyle's law $pv = k$, find the work done by the pressure of the gas in pushing out the piston (Fig. 107d).

Let v be the volume of gas in the cylinder and p the pressure per unit area of the piston. If A is the area of the piston, pA is the total pressure of the gas upon it. If s is the distance the piston moves, the work done is

$$W = \int_{s_1}^{s_2} pA\, ds.$$

But $A\, ds = dv$. Hence

$$W = \int_{v_1}^{v_2} p\, dv = \int_{v_1}^{v_2} \frac{k}{v} dv = k \ln \frac{v_2}{v_1}$$

is the work done when the volume expands from v_1 to v_2.

Example 3. The force with which an electric charge e_1 repels a charge e_2 at distance r is

$$\frac{ke_1e_2}{r^2},$$

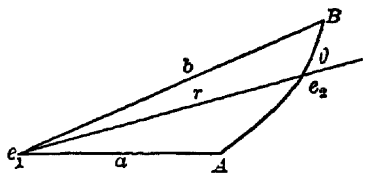

FIG 107e.

where k is constant. Find the work done by this force when the charge e_2 moves from $r = a$ to $r = b$, e_1 remaining fixed.

Let the charge e_2 move from A to B along any path AB (Fig. 107e). The work done by the force of repulsion is

$$W = \int F \cos \theta\, ds = \int F\, dr = \int_a^b \frac{ke_1e_2}{r^2} dr$$
$$= ke_1e_2\left(\frac{1}{a} - \frac{1}{b}\right).$$

The work depends only on the end points A and B and not on the path connecting them.

EXERCISES

1. According to Hooke's law the force required to stretch an elastic rod of natural length a to the length $a + x$ is

$$\frac{kx}{a},$$

where k is constant. Find the work done in stretching the rod from the length a to the length b.

2. A gas is confined in a cylinder by a movable piston. Assuming that the pressure p (pounds per square inch) and the volume v (cubic inches) occupied by the gas satisfy Boyle's law,

$$pv = k,$$

where k is constant, find the work required to compress 100 cu. in. of air at atmospheric pressure (14.7 lbs. per square inch) to a volume of 10 cu. in.

3. When a gas is compressed adiabatically (that is, without receiving or giving out heat), its pressure and volume satisfy the equation

$$pv^{\gamma} = \text{const.},$$

where γ in case of air is about 1.4. Find the work done in the adiabatic compression of 100 cu. in. of air from atmospheric pressure to a volume of 10 cu. in.

4. If A is at an altitude of h ft. above B (but not vertically above B), show that the work done by gravity when a weight of W lbs. is moved along any path from A to B is Wh ft.-lbs.

5. A cylindrical cistern of diameter 4 ft. and depth 8 ft. is full of water. Find the work required to pump the water over the top. Consider the process as equivalent to lifting the water to the top one infinitesimal layer at a time.

6. A particle P is attracted toward a fixed point O with a force equal to kr, where k is constant and r the distance from O to P. Find the work done by this force when the particle moves from a position where $r = a$ to a position where $r = b$.

7. Assuming that the force of gravity on a mass m at distance r from the center of the earth is

$$\frac{km}{r^2},$$

where k is constant, find the work done by gravity on a 10-lb. body when it moves from an indefinitely great distance to the surface of the earth. Use the known force at the surface of the earth to calculate the constant k.

8. Inside the earth a particle is attracted toward the center with a force proportional to its distance from the center. Find the work required to lift a 10-lb. body from the center to the surface of the earth.

9. A vertical shaft is supported by a flat step bearing (Fig. 107f). The frictional force between a small part of the shaft and the surface of the bearing is μP, where P is the pressure between the two and μ is

constant. If the pressure per unit area is the same at all points of the supporting surface, and the weight of the shaft and its load is W, find the work done by the frictional forces during each revolution of the shaft. Divide the surface of the shaft into infinitesimal rings and compute the work done on each ring.

10. When an electric current flows a distance x through a homogeneous conductor of cross section A, the resistance is

$$\frac{kx}{A}$$

where k is a constant depending on the material. Find the resistance when the current flows from the inner to the outer surface of a hollow cylinder, the inner radius being a, the outer radius b, and the altitude h.

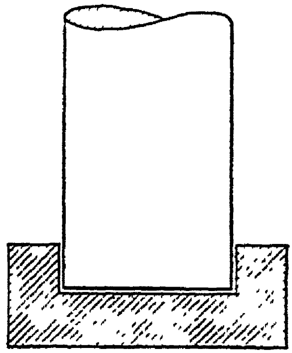

FIG. 107f.

11. Find the resistance when the current flows from the inner to the outer surface of a hollow sphere, the two radii being a and b.

108. Mean Value of a Function. — Between $x = a$ and $x = b$ let

$$y = f(x)$$

be a continuous function of x. Divide the interval $b-a$ into n equal parts

$$\Delta x = \frac{b-a}{n}$$

and let $y_0, y_1, y_2, \cdots, y_{n-1}$ be the values of y at $x = a$ and the points of division. The average value of these n numbers is

$$\frac{y_0 + y_1 + y_2 + \cdots + y_{n-1}}{n}.$$

Multiplying numerator and denominator by Δx, this fraction becomes

$$\frac{y_0\,\Delta x + y_1\,\Delta x + \cdots + y_{n-1}\,\Delta x}{n\,\Delta x} = \frac{\Sigma_a^b y\,\Delta x}{b-a} = \frac{\Sigma_a^b f(x)\,\Delta x}{b-a}.$$

When n is indefinitely increased this average value approaches the limit

$$\frac{1}{b-a}\int_a^b f(x)\,dx, \tag{108}$$

which is called the *mean value of* $f(x)$ *between* $x = a$ *and* $x = b$.

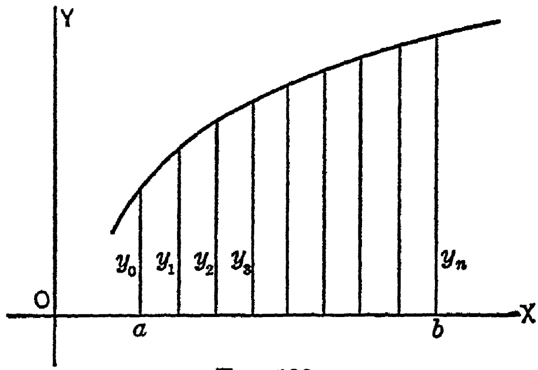

FIG. 108a.

This may be considered as a sort of average of all the ordinates of the curve $y = f(x)$ between $x = a$ and $x = b$ but it should be noted that the result depends on the variable x in terms of which y is expressed. In the above discussion we have taken ordinates at equal intervals along the x-axis and have obtained as final result the mean value of y with respect to x. If y is also expressible in terms of some other variable t and we take ordinates at equal intervals Δt the resulting mean with respect to t need not have the same value.

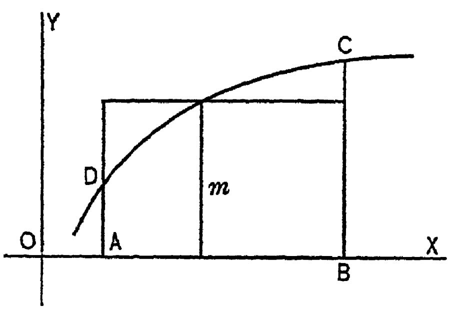

FIG. 108b.

The mean value is graphically represented as the height

$$m = \frac{1}{b-a}\int_a^b f(x)\,dx$$

of the rectangle having the same base and area as the region $ABCD$ bounded by the x-axis, the curve $y = f(x)$ and the ordinates $x = a$, $x = b$.

Example. Find the mean value of the perpendiculars from a diameter of a semicircle to its circumference, assuming the perpendiculars drawn at equal distances along the circumference.

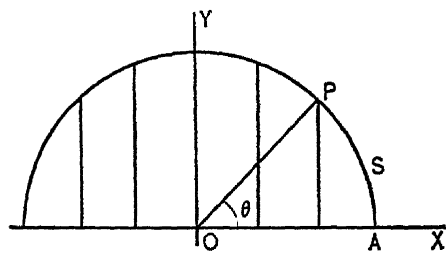

FIG. 108c.

Let s be the arc from A (Fig. 108c) to the variable point P on the circumference. Since the perpendiculars are taken at equal intervals along the circumference, we must average with respect to s. The mean value required is therefore

$$\frac{1}{\pi a - 0}\int_0^{\pi a} y\, ds,$$

where a is the radius. If θ is the angle subtended at the center by the arc s,

$$y = a \sin\theta, \qquad s = a\theta.$$

Hence the above expression is equivalent to

$$\frac{a}{\pi}\int_0^{\pi} \sin\theta\, d\theta = \frac{2\,a}{\pi}.$$

109. Theorems of Pappus. — Theorem I. — *If the arc of a plane curve is revolved about an axis in its plane and not crossing the arc, the area generated is equal to the product of the length of the arc and the length of the path described by its center of gravity.*

Let the arc be rotated about the x-axis. The ordinate of its center of gravity is

$$\bar{y} = \frac{\int y\, ds}{s},$$

where s is the length of arc. Consequently

$$2\,\pi\bar{y}s = \int 2\,\pi y\,ds.$$

The right side of this equation represents the area generated. Also $2\,\pi\bar{y}$ is the length of the path described by the center of gravity. This equation therefore expresses the result to be proved.

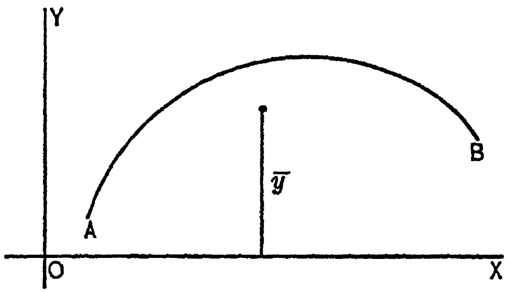

FIG 109a.

The area can be considered as a sum of strips each generated by a small piece of the arc. The area of such a strip is the product of its width and circumference. The theorem states that the average circumference of the strips is that described by the center of gravity of the arc.

Theorem II. *If a plane area is revolved about an axis in its plane and not crossing the area, the volume generated is equal to the product of the area and the length of the path described by its center of gravity.*

Let the area be revolved about the x-axis. Cut the area A into strips dA parallel to the x-axis. The ordinate of the center of gravity is

$$\bar{y} = \frac{\int y\,dA}{A},$$

whence

$$2\,\pi\bar{y}A = \int 2\,\pi y\,dA.$$

The right side of this equation represents the volume generated. Also $2\pi\bar{y}$ is the length of the path described by the center of gravity. This equation therefore expresses the result to be proved.

The volume may be considered as a sum of rings each generated by a small piece of the area A. The volume of any one ring is the product of its circumference and area of section. The theorem expresses that the mean or average circumference of all the rings is that described by the center of gravity.

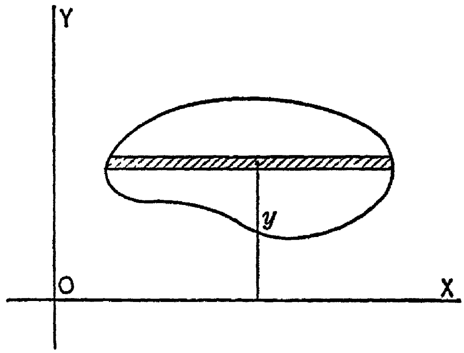

FIG 109b.

Example 1. Find the area of the torus generated by revolving a circle of radius a about an axis in its plane at distance b (greater than a) from its center.

Since the circumference of the circle is $2\pi a$ and the length of the path described by its center $2\pi b$, the area generated is

$$S = 2\pi a \cdot 2\pi b = 4\pi^2 ab.$$

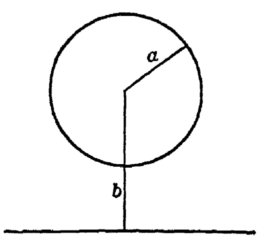

FIG. 109c.

Example 2. Find the center of gravity of the area of a semicircle by using Pappus's theorems.

When a semicircle of radius a is revolved about its diameter, the volume of the sphere generated is $\frac{4}{3}\pi a^3$. If $\bar{y}$ is the distance of the center of gravity of the semicircle from this diameter, by the first theorem of Pappus,

$$\tfrac{4}{3}\pi a^3 = 2\pi\bar{y}A = 2\pi\bar{y}\cdot\tfrac{1}{2}\pi a^2,$$

whence

$$\bar{y} = \frac{\frac{4}{3}\pi a^3}{\pi^2 a^2} = \frac{4a}{3\pi}.$$

Example 3. Find the volume generated by revolving the cardioid $r = a\,(1 + \cos\theta)$ about the initial line.

The area of the triangle OPQ is approximately

$$\tfrac{1}{2}\,r^2\,\Delta\theta,$$

and its center of gravity is $\tfrac{2}{3}$ of the distance from the vertex to the base. Hence

$$\bar{y} = \tfrac{2}{3}\,r\sin\theta.$$

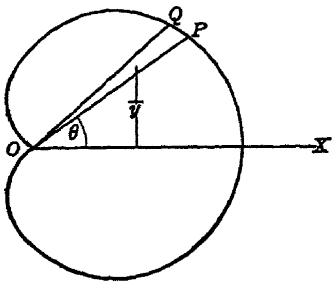

FIG. 109d.

By the second theorem of Pappus, the volume generated by OPQ is then approximately

$$2\,\pi\bar{y}\,\Delta A = \tfrac{2}{3}\,\pi r^3\sin\theta\,\Delta\theta.$$

The entire volume is therefore

$$v = \int_0^\pi \tfrac{2}{3}\,\pi r^3 \sin\theta\,d\theta = \tfrac{2}{3}\,\pi a^3\int_0^\pi (1 + \cos\theta)^3\sin\theta\,d\theta$$

$$= \left[-\frac{2}{3}\pi a^3\frac{(1 + \cos\theta)^4}{4}\right]_0^\pi = \frac{8}{3}\pi a^3.$$

EXERCISES

1. Find the mean value of the lengths of the perpendiculars from the diameter of a semicircle to its circumference, assuming the perpendiculars drawn at equal distances along the diameter.

2. Find the mean value of the ordinates $y = \sin x$ between $x = 0$ and $x = \pi$ if the ordinates are taken at equal distances along the x-axis.

3. A body falls from rest. Find its mean velocity during the first t seconds, if the average is taken with respect to the time. Show that the distance it falls is equal to this mean velocity multiplied by the time.

4. A body falls from rest. Find the mean velocity during the first t seconds if the average is taken with respect to the distance. Use the formula $s = \frac{1}{2}\,gt^2$ to express the integral in terms of the time.

5. A cubic foot of air is compressed from atmospheric pressure (14.7 lbs. per sq. in.) to one-tenth its initial volume. If the gas satisfies Boyle's law $pv = k$, find the mean pressure, the average being taken with

respect to the volume. Show that the work required to compress the gas is this mean pressure multiplied by the change of volume.

6. A current of i amperes flowing through a resistance of R ohms produces heat at the rate of Ri^2 joules per second. Find the average rate at which heat is produced by an alternating current

$$i = A \sin (\omega t)$$

during one cycle $\left(t = 0 \text{ to } t = \frac{2\pi}{\omega}\right)$. Find the steady current I which produces the same total heat during one cycle.

7. A body is cut into a large number of small particles of equal mass. When the number of particles is increased indefinitely show that their average distance from any plane approaches the distance of the center of gravity from that plane.

8. Let b (greater than a) be the distance from a point Q to the center of a sphere of radius a and r the distance from Q to a variable point P on the surface of the sphere. Find the average value of

$$\frac{1}{r}$$

if the points P are spread uniformly over the surface of the sphere.

9. By using the theorems of Pappus find the lateral area and the volume of a right circular cone

10. Find the volume of the torus generated by revolving a circle of radius a about an axis in its plane at distance b (greater than a) from its center.

11. A groove with cross section an equilateral triangle of side $\frac{1}{2}$ inch is cut around a cylindrical shaft 6 inches in diameter. Find the volume of material cut away.

12. The length of an arch of the cycloid

$$x = a(\phi - \sin \phi), \qquad y = a(1 - \cos \phi)$$

is $8\,a$ and the area generated by revolving it around the x-axis is $\frac{64}{3}\pi a^2$. Find the area generated by revolving the arch about the tangent at its highest point.

13. Show that the pressure on a submerged plane area is equal to the product of its area by the pressure at its center of gravity.

14. By the method of Ex. 3, page 208, find the volume generated by rotating a circular sector of radius a and central angle α about one of its bounding radii.

15. By the method of the preceding problem find the volume generated by rotating the cardioid

$$r = a(1 + \sin \theta)$$

about the initial line.

16. Find the center of gravity of the volume in Ex. 14. Use the hollow conical slices generated by the polar elements of area and the fact that the center of gravity of a cone is three-fourths of the way from its vertex to its base.

CHAPTER XVI

PARTIAL DIFFERENTIATION

110. Functions of Two or More Variables. — A quantity u is called a function of two independent variables x and y,

$$u = f(x, y),$$

if u is determined when arbitrary values (or values arbitrary within certain limits) are assigned to x and y.

For example,

$$u = \sqrt{1 - x^2 - y^2}$$

is a function of x and y. If u is to be real, x and y must be so chosen that $x^2 + y^2$ is not greater than 1. Within that limit, however, x and y can be chosen independently and a value of u will then be determined.

In a similar way we define a function of three or more independent variables. An illustration of a function of variables that are not independent is furnished by the area of a triangle. It is a function of the sides a, b, c and angles A, B, C of the triangle, but is not a function of these six quantities considered as independent variables; for, if values not belonging to the same triangle are given to them, no triangle and consequently no area will be determined.

The increment of a function of several variables is its increase when all the variables change. Thus, if

$$u = f(x, y),$$
$$u + \Delta u = f(x + \Delta x, y + \Delta y)$$

and so

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y).$$

A function is called *continuous* if its increment approaches zero when the increments of all the variables approach zero.

111. Partial Derivatives. — Let

$$u = f(x, y)$$

be a function of two independent variables x and y. If we keep y constant, u is a function of x. The derivative of this function with respect to x is called the *partial derivative* of u with respect to x and is denoted by

$$\frac{\partial u}{\partial x} \quad \text{or} \quad f_x(x, y).$$

Similarly, if we differentiate with respect to y with x constant, we get the partial derivative with respect to y denoted by

$$\frac{\partial u}{\partial y} \quad \text{or} \quad f_y(x, y).$$

For example, if

$$u = x^2 + xy - y^2,$$

then

$$\frac{\partial u}{\partial x} = 2x + y, \quad \frac{\partial u}{\partial y} = x - 2y.$$

Likewise, if u is a function of any number of independent variables, the partial derivative with respect to one of them is obtained by differentiating with the others constant.

112. Higher Derivatives. — The first partial derivatives are functions of the variables. By differentiating these functions partially, we get higher partial derivatives.

For example, the derivatives of $\frac{\partial u}{\partial x}$ with respect to x and y are

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\, \partial x}.$$

Similarly,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\, \partial y}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}.$$

It can be shown that

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x},$$

if both derivatives are continuous, that is, *partial derivatives are independent of the order in which the differentiations are performed.**

Example. $u = x^2y + xy^2$.

$$\frac{\partial u}{\partial x} = 2\,xy + y^2, \quad \frac{\partial u}{\partial y} = x^2 + 2\,xy,$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(2\,xy + y^2) = 2\,y, \quad \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}(2\,xy + y^2) = 2\,x + 2\,y,$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}(x^2 + 2\,xy) = 2\,x + 2\,y, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(x^2 + 2\,xy) = 2\,x.$$

113. Dependent Variables. — It often happens that some of the variables are functions of others. For example, let

$$u = x^2 + y^2 + z^2$$

and let z be a function of x and y. When y is constant, z will be a function of x and the partial derivative of u with respect to x will be

$$\frac{\partial u}{\partial x} = 2\,x + 2\,z\frac{\partial z}{\partial x}.$$

Similarly, the partial derivative with respect to y with x constant is

$$\frac{\partial u}{\partial y} = 2\,y + 2\,z\frac{\partial z}{\partial y}.$$

If, however, we consider z constant, the partial derivatives are

$$\frac{\partial u}{\partial x} = 2\,x, \quad \frac{\partial u}{\partial y} = 2\,y.$$

The value of a partial derivative thus depends on what quantities are kept constant during the differentiation.

The quantities kept constant are sometimes indicated by subscripts. Thus, in the above example

$$\left(\frac{\partial u}{\partial x}\right)_{y,z} = 2\,x, \quad \left(\frac{\partial u}{\partial x}\right)_y = 2\,x + 2\,z\frac{\partial z}{\partial x}, \quad \left(\frac{\partial u}{\partial x}\right)_z = 2\,x + 2\,y\frac{\partial y}{\partial x}.$$

* For a proof see Wilson, *Advanced Calculus*, § 50.

It will usually be clear from the context what independent variables u is considered a function of. Then $\frac{\partial u}{\partial x}$ will represent the derivative with all those variables except x constant.

Example. If a is a side and A the opposite angle of a right triangle with hypotenuse c, find $\left(\frac{\partial a}{\partial c}\right)_A$.

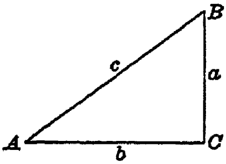

FIG. 113.

From the triangle it is seen that

$$a = c \sin A.$$

Differentiating with A constant, we get

$$\frac{\partial a}{\partial c} = \sin A,$$

which is the value required.

114. Geometrical Representation. — Let $z = f(x, y)$ be the equation of a surface. The points with constant y-coördinate form the curve AB (Fig. 114a) in which the plane $y =$ constant intersects the surface. In this plane z is the vertical and x the horizontal coördinate. Consequently,

$$\frac{\partial z}{\partial x}$$

is the slope of the curve AB at P.

Similarly, the locus of points with given x is the curve CD and

$$\frac{\partial z}{\partial y}$$

is the slope of this curve at P.

Example. Find the lowest point on the paraboloid

$$z = x^2 + y^2 - 2x - 4y + 6.$$

At the lowest point, the curves AB and CD (Fig. 114b) will have horizontal tangents. Hence

$$\frac{\partial z}{\partial x} = 2x - 2 = 0, \qquad \frac{\partial z}{\partial y} = 2y - 4 = 0.$$

Consequently, $x = 1$, $y = 2$. These values substituted in the equation of the surface give $z = 1$. The point required is then (1, 2, 1). That this is really the lowest point is shown by the graph.

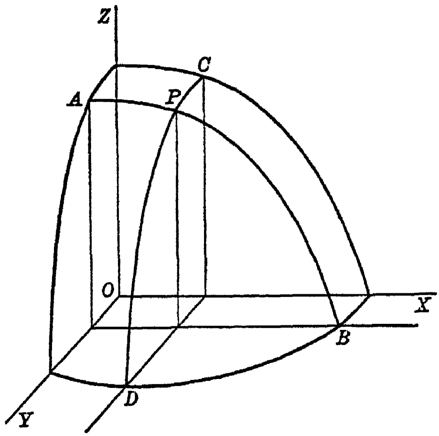

FIG. 114a.

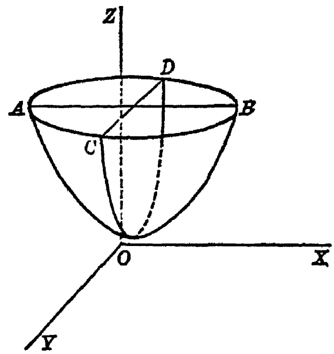

FIG. 114b.

EXERCISES

In each of the following exercises show that the partial derivatives satisfy the equation given:

1. $u = \dfrac{x^2 + y^2}{x + y}$, $\quad x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = u.$

2. $z = (x + a)(y + b)$, $\quad \dfrac{\partial z}{\partial x} \dfrac{\partial z}{\partial y} = z.$

3. $z = (x^2 + y^2)^n$, $\quad y \dfrac{\partial z}{\partial x} = x \dfrac{\partial z}{\partial y}.$

4. $u = \ln(x^2 + xy + y^2)$, $\quad x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 2.$

5. $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$, $\quad x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} = 0.$

6. $u = \tan^{-1}\left(\dfrac{y}{x}\right)$, $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

7. $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, $\quad \dfrac{\partial^2 u}{dx^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$

In each of the following exercises verify that

$$\frac{\partial^2 u}{\partial x \, \partial y} = \frac{\partial^2 u}{\partial y \, \partial x}.$$

8. $u = \frac{y}{x}$.

9. $u = \ln (x^2 + y^2)$.

10. $u = \sin (x + y)$.

11. $u = xyz$.

12. Given $v = \sqrt{x^2 + y^2 + z^2}$, verify that

$$\frac{\partial^3 v}{\partial x\, \partial y\, \partial z} = \frac{\partial^3 v}{\partial z\, \partial y\, \partial x}.$$

Prove the following relations assuming that z is a function of x and y:

13. $u = (x + z)\, e^{y+z}$, $\quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = (1 + x + z)\left(1 + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) e^{y+z}$.

14. $u = xyz$, $\quad z\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right) = u\left(x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}\right)$.

15. $u = e^x + e^y + e^z$, $\quad \frac{\partial^2 u}{\partial x\, \partial y} = e^z\left(\frac{\partial^2 z}{\partial x\, \partial y} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right)$.

16. $\frac{\partial}{\partial x}\left(z\frac{\partial u}{\partial x} - u\frac{\partial z}{\partial x}\right) = z\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^2 z}{\partial x^2}$.

17. If $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y.$$

18. Let a and b be the sides of a right triangle with hypotenuse c and opposite angles A and B. Let p be the perpendicular from the vertex of the right angle to the hypotenuse. Show that

$$\left(\frac{\partial p}{\partial a}\right)_b = \frac{b^3}{c^3}, \qquad \left(\frac{\partial p}{\partial a}\right)_A = \frac{b}{c}.$$

19. If K is the area of a triangle, a side and two adjacent angles of which are c, A, B, show that

$$\left(\frac{\partial K}{\partial A}\right)_{c,B} = \frac{b^2}{2}, \qquad \left(\frac{\partial K}{\partial B}\right)_{c,A} = \frac{a^2}{2}.$$

20. If K is the area of a triangle with sides a, b, c, show that

$$\left(\frac{\partial K}{\partial a}\right)_{b,c} = \frac{a}{2}\cot A.$$

21. Find the lowest point on the surface

$$z = 2x^2 + y^2 + 8x - 2y + 9.$$

22. Find the highest point on the surface

$$z = 2y - x^2 + 2xy - 2y^2 + 1.$$

115. Increment. — Let $u = f(x, y)$ be a function of two independent variables x and y. When x changes to $x + \Delta x$ and y to $y + \Delta y$, the increment of u is

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y). \qquad (115a)$$

By the mean value theorem, Art. 146,

$$f(x + \Delta x, y + \Delta y) = f(x, y + \Delta y) + \Delta x f_x(x_1, y + \Delta y),$$

x_1 lying between x and $x + \Delta x$. Similarly

$$f(x, y + \Delta y) = f(x, y) + \Delta y f_y(x, y_1),$$

y_1 being between y and $y + \Delta y$. Using these values in (115a), we get

$$\Delta u = \Delta x f_x(x_1, y + \Delta y) + \Delta y f_y(x, y_1). \qquad (115b)$$

As Δx and Δy approach zero, x_1 approaches x and y_1 approaches y. If $f_x(x, y)$ and $f_y(x, y)$ are continuous,

$$f_x(x_1, y + \Delta y) = f_x(x, y) + \epsilon_1 = \frac{\partial u}{\partial x} + \epsilon_1,$$

$$f_y(x, y_1) = f_y(x, y) + \epsilon_2 = \frac{\partial u}{\partial y} + \epsilon_2,$$

ϵ_1 and ϵ_2 approaching zero as Δx and Δy approach zero. These values substituted in (115b) give

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y. \qquad (115c)$$

The quantity

$$\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y$$

is called the *principal part* of Δu. It differs from Δu by an amount $\epsilon_1 \Delta x + \epsilon_2 \Delta y$. As Δx and Δy approach zero, ϵ_1 and ϵ_2 approach zero and so this difference becomes an indefinitely small fraction of the larger of the increments Δx and Δy. We express this by saying the principal part differs from Δu by an infinitesimal of higher order than Δx and Δy (Art. 46). When Δx and Δy are sufficiently small this principal part then gives a satisfactory approximation for Δu.

Analogous results can be obtained for any number of independent variables. For example, if there are three independent variables x, y, z, the principal part of Δu is

$$\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z.$$

In each case, if the partial derivatives are continuous, the

principal part differs from Δu by an amount which becomes indefinitely small in comparison with the largest of the increments of the independent variables as those increments all approach zero.

Example. Find the change in the volume of a cylinder when its length increases from 6 ft. to 6 ft. 1 in. and its diameter decreases from 2 ft. to 23 in.

Since the volume is $v = \pi r^2 h$, the exact change is

$$\Delta v = \pi \left(1 - \tfrac{1}{24}\right)^2 \left(6 + \tfrac{1}{12}\right) - \pi \cdot 1^2 \cdot 6 = -0.413\,\pi \text{ cu. ft.}$$

The principal part of this increment is

$$\frac{\partial v}{\partial r}\,\Delta r + \frac{\partial v}{\partial h}\,\Delta h = 2\,\pi\,rh\left(-\frac{1}{24}\right) + \pi\,r^2\left(\frac{1}{12}\right) = -0.417\,\pi \text{ cu ft.}$$

116. Total Differential. — If u is a function of two independent variables x and y, the *total differential* of u is the principal part of Δu, that is,

$$du = \frac{\partial u}{\partial x}\,\Delta x + \frac{\partial u}{\partial y}\,\Delta y. \tag{116a}$$

This definition applies to any function of x and y. The particular values $u = x$ and $u = y$ give

$$dx = \Delta x, \qquad dy = \Delta y; \tag{116b}$$

that is, *the differentials of the independent variables are equal to their increments.*

Combining (116a) and (116b), we get

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy. \tag{116c}$$

We shall show later (Art. 120) that this equation is valid even if x and y are not the independent variables.

The quantities

$$d_x u = \frac{\partial u}{\partial x}\,dx, \qquad d_y u = \frac{\partial u}{\partial y}\,dy$$

are called *partial differentials.* Equation (116c) expresses that *the total differential of a function is equal to the sum of the partial differentials obtained by letting the variables change one at a time.*

Similar results can be obtained for functions of any number of variables. For instance, if u is a function of three independent variables x, y, z,

$$du = \frac{\partial u}{\partial x}\,\Delta x + \frac{\partial u}{\partial y}\,\Delta y + \frac{\partial u}{\partial z}\,\Delta z.$$

The particular values $u = x,\ u = y,\ u = z$ give

$$dx = \Delta x, \qquad dy = \Delta y, \qquad dz = \Delta z.$$

The previous equation can then be written

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz \tag{116d}$$

and in this form it can be proved valid even when x, y, z are not the independent variables.

Example 1. Find the total differential of the function

$$u = x^2y + xy^2.$$

By equation (116c)

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ &= (2\,xy + y^2)\,dx + (x^2 + 2\,xy)\,dy. \end{aligned}$$

Ex. 2. Find the error in the volume of a rectangular box due to small errors in its three edges.

Let the edges be x, y, z. The volume is then

$$v = xyz.$$

The error in v, due to small errors $\Delta x, \Delta y, \Delta z$ in x, y, z, is Δv. If the increments are sufficiently small, this will be approximately

$$dv = yz\,dx + xz\,dy + xy\,dz.$$

Dividing by v, we get

$$\begin{aligned} \frac{dv}{v} &= \frac{yz\,dx + xz\,dy + xy\,dz}{xyz} \\ &= \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}. \end{aligned}$$

Now $\frac{dx}{x}$ expresses the error dx as a fraction or percentage of x.

The equation just obtained expresses that the percentage error in the volume is equal to the sum of the percentage errors in the edges. If, for example, the error in each edge is not more than one per cent, the error in the volume is not more than three per cent.

117. Calculation of Differentials. — In proving the formulas of differentiation it was assumed that u, v, etc., were functions of a single variable. It is easy to show that the same formulas are valid when those quantities are functions of two or more variables and du, dv, etc., are their total differentials.

Take, for example, the differential of uv. By (116c) the result is

$$d(uv) = \frac{\partial}{\partial u}(uv)\,du + \frac{\partial}{\partial v}(uv)\,dv = v\,du + u\,dv,$$

which is the formula IV of Art. 50.

Example. $u = ye^x + ze^y$.

Differentiating term by term, we get

$$du = ye^x\,dx + e^x\,dy + ze^y\,dy + e^y\,dz.$$

We obtain the same result by using (116d); for that formula gives

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = ye^x\,dx + (e^x + ze^y)\,dy + e^y\,dz.$$

118. Partial Derivatives as Ratios of Differentials. — The equation

$$d_x u = \frac{\partial u}{\partial x}dx$$

shows that the partial derivative $\frac{\partial u}{\partial x}$ is the ratio of two differentials $d_x u$ and dx. Now $d_x u$ is the value of du when the same quantities are kept constant that are constant in the calculation of $\frac{\partial u}{\partial x}$. Therefore, *the partial derivative* $\frac{\partial u}{\partial x}$ *is the*

value to which $\frac{du}{dx}$ *reduces when du and dx are determined with the same quantities constant that are constant in the calculation of* $\frac{\partial u}{\partial x}$.

Example. Given $u = x^2 + y^2 + z^2$, $v = xyz$, find $\left(\frac{\partial u}{\partial x}\right)_{v,z}$.

Differentiating the two equations with v and z constant, we get

$$du = 2\,x\,dx + 2\,y\,dy, \qquad 0 = yz\,dx + xz\,dy.$$

Eliminating dy,

$$du = 2\,x\,dx - 2\,\frac{y^2}{x}\,dx = 2\left(\frac{x^2 - y^2}{x}\right)dx.$$

Under the given conditions the ratio of du to dx is then

$$\frac{du}{dx} = \frac{2\,(x^2 - y^2)}{x}.$$

Since v and z were kept constant, this ratio represents $\left(\frac{\partial u}{\partial x}\right)_{v,z}$; that is,

$$\left(\frac{\partial u}{\partial x}\right)_{v,z} = \frac{2\,(x^2 - y^2)}{x}.$$

EXERCISES

1. The sides of a rectangle are $x = 10$, $y = 12$. If each of these is increased one unit, find the increment of the area and its principal part.

2. One side of a right triangle increases from 5 to 5 2 while the other decreases from 12 to 11.5. Find the increment of the hypotenuse and its principal part.

3. A box is 12 in. long, 8 in wide, and 6 in deep. If each edge is increased $\frac{1}{4}$ in., find the increment of the volume and its principal part.

4. Two sides and the included angle of a triangle are $b = 20$, $c = 30$, $A = 60°$. By using the formula

$$a^2 = b^2 + c^2 - 2\,bc\cos A,$$

find approximately the change in a when b increases one unit, c decreases $\frac{1}{2}$ unit, and A increases 1 degree.

5. The period of a simple pendulum is

$$T = 2\,\pi\sqrt{\frac{l}{g}}.$$

Find the maximum error in T due to errors of 1% in l and g.

6. If g is computed by the formula

$$s = \tfrac{1}{2} gt^2,$$

find the maximum error in g due to errors of 1% in s and t.

7. The area of a triangle is calculated by the formula

$$K = \tfrac{1}{2} ab \sin C.$$

If C is approximately 30°, find the maximum error in K due to errors of 0.1% in a and b and 10 minutes in C.

Find the total differentials of the following functions:

8. $x^2 + xy - y^2$.
9. $xy \sin (x + y)$.
10. $xy^2 z^3$.
11. $\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$.

12. If x, y are rectangular and r, θ polar coördinates of the same point, show that

$$x\,dy - y\,dx = r^2\,d\theta, \qquad dx^2 + dy^2 = dr^2 + r^2\,d\theta^2.$$

13. If $x = u + v$, $y = u - v$, show that

$$\left(\frac{\partial y}{\partial x}\right)_u = -\left(\frac{\partial y}{\partial x}\right)_v.$$

14. If $x = \dfrac{r}{2}(e^\theta + e^{-\theta})$, $y = \dfrac{r}{2}(e^\theta - e^{-\theta})$, show that

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y.$$

119. Derivative of a Function of Several Variables. — Let $u = f(x, y)$ and let x and y be functions of two variables s and t. When t changes to $t + \Delta t$, x and y will change to $x + \Delta x$ and $y + \Delta y$. The resulting increment in u will be

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y.$$

Consequently,

$$\frac{\Delta u}{\Delta t} = \frac{\partial u}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t}.$$

As Δt approaches zero, Δx and Δy will approach zero and so ϵ_1 and ϵ_2 will approach zero. Taking the limit of both sides,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}. \tag{119a}$$

If x or y is a function of t only, the partial derivative $\frac{\partial x}{\partial t}$ or $\frac{\partial y}{\partial t}$ is replaced by a total derivative $\frac{dx}{dt}$ or $\frac{dy}{dt}$. If both x and y are functions of t, u is a function of t with total derivative

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}. \tag{119b}$$

Likewise, if u is a function of three variables x, y, z, that depend on t,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}. \tag{119c}$$

As before, if a variable is a function of t only, its partial derivative is replaced by a total one. Similar results hold for any number of variables.

The term

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}$$

is the result of differentiating u with respect to t, leaving all the variables in u except x constant. Equations (119a) and (119c) express that *if u is a function of several variable quantities, $\frac{\partial u}{\partial t}$ can be obtained by differentiating with respect to t as if only one of those quantities were variable at a time and adding the results.*

Example 1. Given $y = x^x$, find $\frac{dy}{dx}$.

The function x^x can be considered a function of two variables, the lower x and the upper x. If the upper x is held constant and the lower allowed to vary, the derivative (as in case of x^n) is

$$x \cdot x^{x-1} = x^x.$$

If the lower x is held constant while the upper varies, the derivative (as in case of a^x) is

$$x^x \ln x.$$

The actual derivative of y is then the sum

$$\frac{dy}{dx} = x^x + x^x \ln x.$$

Ex. 2. Given $u = f(x, y, z)$, y and z being functions of x, find $\frac{du}{dx}$.

By equation (119c) the result is

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}.$$

In this equation there are two derivatives of u with respect to x. If y and z are replaced by their values in terms of x, u will be a function of x only. The derivative of that function is $\frac{du}{dx}$. If y and z are replaced by constants, u will be a second function of x. Its derivative is $\frac{\partial u}{\partial x}$.

Ex. 3. Given $u = f(x, y, z)$, z being a function of x and y. Find the partial derivative of u with respect to x.

It is understood that y is to be constant in this partial differentiation. Equation (119c) then gives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}.$$

In this equation appear two partial derivatives of u with respect to x. If z is replaced by its value in terms of x and y, u will be expressed as a function of x and y only. Its partial derivative is the one on the left side of the equation. If z is kept constant, u is again a function of x and y. Its partial derivative appears on the right side of the equation. We must not of course use the same symbol for both of these derivatives. A way to avoid the confusion is to use the

letter f instead of u on the right side of the equation. It then becomes

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}.$$

It is understood that $f(x, y, z)$ is a definite function of x, y, z and that $\frac{\partial f}{\partial x}$ is the derivative obtained with all the variables but x constant.

120. Change of Variable. — If u is a function of x and y we have said that the equation

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

is true whether x and y are the independent variables or not. To show this let s and t be the independent variables and x and y functions of them. Then, by definition,

$$du=\frac{\partial u}{\partial s}ds+\frac{\partial u}{\partial t}dt.$$

Since u is a function of x and y which are functions of s and t, by equation (119a),

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s},\qquad \frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}.$$

Consequently,

$$\begin{aligned}du&=\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\right)ds+\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right)dt\\&=\frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial s}ds+\frac{\partial x}{\partial t}dt\right)+\frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial s}ds+\frac{\partial y}{\partial t}dt\right)=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy,\end{aligned}$$

which was to be proved.

A similar proof can be given in case of three or more variables.

121. Implicit Functions. — If two or more variables are connected by an equation, a differential relation can be obtained by equating the total differentials of the two sides of the equation.

Example 1. $f(x, y) = 0$.

In this case

$$d \cdot f(x, y) = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = d \cdot 0 = 0.$$

Consequently,

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}.$$

Ex. 2. $f(x, y, z) = 0$.

Differentiation gives

$$\frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz = 0.$$

If z is considered a function of x and y, its partial derivative with respect to x is found by keeping y constant. Then $dy = 0$ and

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial z}}.$$

Similarly, if x is constant, $dx = 0$ and

$$\frac{\partial z}{\partial y} = -\frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial z}}.$$

Ex. 3. $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$.

We have two differential relations

$$\frac{\partial f_1}{\partial x}\, dx + \frac{\partial f_1}{\partial y}\, dy + \frac{\partial f_1}{\partial z}\, dz = 0,$$
$$\frac{\partial f_2}{\partial x}\, dx + \frac{\partial f_2}{\partial y}\, dy + \frac{\partial f_2}{\partial z}\, dz = 0.$$

We could eliminate y from the two equations $f_1 = 0$, $f_2 = 0$. We should then obtain z as a function of x. The total de-

rivative of this function is found by eliminating dy and solving for the ratio $\frac{dz}{dx}$. The result is

$$\frac{dz}{dx} = \frac{\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y}}{\frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z}}.$$

122. Exact Differentials. — If P and Q are functions of two independent variables x and y,

$$P\,dx + Q\,dy$$

may or may not be the total differential of a function u of x and y. If it is the total differential of such a function,

$$P\,dx + Q\,dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

Since dx and dy are arbitrary, this requires

$$P = \frac{\partial u}{\partial x}, \qquad Q = \frac{\partial u}{\partial y}.$$

Consequently,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{dy\,\partial x}, \qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

Since the two second derivatives of u with respect to x and y are equal,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{122a}$$

An expression $P\,dx + Q\,dy$ is called an *exact differential* if it is the total differential of a function of x and y. We have just shown that (122a) must then be satisfied. Conversely, it can be shown that if this equation is satisfied $P\,dx + Q\,dy$ is an exact differential.*

* See Wilson, *Advanced Calculus*, § 92.

Similarly, if

$$P\,dx + Q\,dy + R\,dz$$

is the differential of a function u of x, y, z,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \qquad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \qquad (122b)$$

and conversely.

Example 1. Show that

$$(x^2 + 2\,xy)\,dx + (x^2 + y^2)\,dy$$

is an exact differential.

In this case

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2\,xy) = 2\,x, \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2\,x.$$

The two partial derivatives being equal, the expression is exact.

Example 2. In thermodynamics it is shown that

$$dU = T\,dS - p\,dv,$$

U being the internal energy, T the absolute temperature, S the entropy, p the pressure, and v the volume of a homogeneous substance. Any two of these five quantities can be assigned independently and the others are then determined. Show that

$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial v}{\partial S}\right)_p.$$

The result to be proved expresses that

$$T\,dS + v\,dp$$

is an exact differential. That such is the case is shown by replacing $T\,dS$ by its value $dU + p\,dv$. We thus get

$$T\,dS + v\,dp = dU + p\,dv + v\,dp = d(U + pv).$$

EXERCISES

1. If $u = f(x, y)$, $y = x$, find $\dfrac{du}{dx}$.

2. If $u = f(x, y, z)$, $z = x^2$, find $\left(\dfrac{\partial u}{\partial x}\right)_y$.

3. If $u = f(x, y, z)$, $y = x$, $z = 2x$, find $\dfrac{du}{dx}$.

4. If $u = f(x, y)$, $y = x + r$, $r = x - s$, find $\left(\dfrac{\partial u}{\partial x}\right)_y$, $\left(\dfrac{\partial u}{\partial x}\right)_r$, $\left(\dfrac{\partial u}{\partial x}\right)_s$.

5. If $f(x, y, z) = 0$, $z = x + y$, find $\dfrac{dz}{dx}$.

6. If $F(x, y, z) = 0$, show that

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

7. If $u = xf(z)$, $z = \dfrac{y}{x}$, and x, y are the independent variables, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

8. If $u = f(r, s)$, $r = x + at$, $s = y + bt$, and x, y, t are the independent variables, show that

$$\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}.$$

9. If $u = f(x + ay)$, show that

$$\frac{\partial u}{\partial y} = a\frac{\partial u}{\partial x}.$$

10. If the substitution $x = r\cos\theta$, $y = r\sin\theta$ changes $f(x, y)$ into $F(r, \theta)$, show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial F}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right)^2.$$

Determine which of the following expressions are exact differentials:

11. $x\,dy - y\,dx$.

12. $(2x + y)\,dx + (x - 2y)\,dy$.

13. $z\,dx + 2yz\,dy + (x + y^2)\,dz$.

14. $x\,dx - (y + z)\,dy + y\,dz$.

15. Under the conditions of Ex. 2, page 228, show that

$$\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial S}{\partial p}\right)_T, \quad \left(\frac{\partial p}{\partial T}\right)_v = \left(\frac{\partial S}{\partial v}\right)_T.$$

16. In case of a perfect gas $pv = kT$. Using this and the equation

$$du = T\,dS - p\,dv,$$

show that

$$\left(\frac{\partial u}{\partial p}\right)_T = 0.$$

This expresses that u is a function of T only.

123. Angle Between Two Directed Lines in Space. — A directed line is one along which a positive direction is assigned. This direction is usually indicated by an arrow. In case of a coördinate axis the positive direction is that in which the coördinate increases.

The angle between two directed lines (Fig. 123) is one along the sides of which the arrows point away from the vertex. There are two such angles less than 360°, their sum being 360°. They have the same cosine.

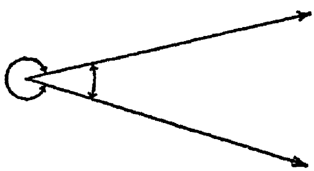

FIG. 123.

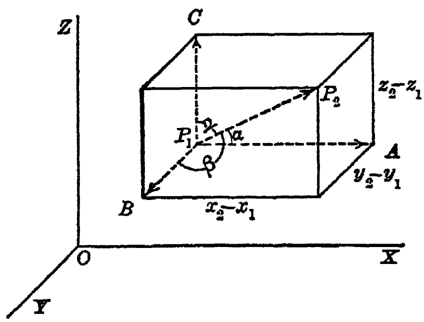

FIG. 124.

If two lines do not intersect, the angle between them is defined as that between lines parallel to them which do intersect.

124. Direction Cosines. — Let α, β, γ be the angles between the coördinate axes and a directed line. The cosines of these angles are called the *direction cosines* of the line.

Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ be two points on the line. Construct a rectangular parallelopiped with diagonal P_1P_2

and edges parallel to the coördinate axes (Fig. 124). Since P_1AP_2, P_1BP_2, and P_1CP_2 are right angles, the direction cosines of P_1P_2 are

$$\cos\alpha = \frac{P_1A}{P_1P_2} = \frac{x_2 - x_1}{P_1P_2},$$

$$\cos\beta = \frac{P_1B}{P_1P_2} = \frac{y_2 - y_1}{P_1P_2},$$

$$\cos\gamma = \frac{P_1C}{P_1P_2} = \frac{z_2 - z_1}{P_1P_2}.$$

Since

$$\overline{P_1P_2}^2 = \overline{P_1A}^2 + \overline{P_1B}^2 + \overline{P_1C}^2,$$

the direction cosines satisfy the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{124a}$$

If the direction cosines of two lines are $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$ and $\cos\alpha_2$, $\cos\beta_2$, $\cos\gamma_2$, it is shown in analytical geometry that the angle θ between the lines is given by the equation

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2. \tag{124b}$$

In particular, if the lines are perpendicular,

$$0 = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2. \tag{124c}$$

125. Direction of the Tangent Line to a Curve. — The tangent line at a point P of a curve is defined as the limiting position PT approached by the secant PQ as Q approaches P along the curve.

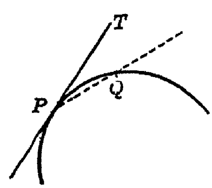

FIG. 125a.

Let s be the arc of the curve measured from some fixed point and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ the direction cosines of the tangent drawn in the direction of increasing s.

If x, y, z are the coördinates of P, $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, those of Q, the direction cosines of PQ are

$$\frac{\Delta x}{\overline{PQ}}, \quad \frac{\Delta y}{\overline{PQ}}, \quad \frac{\Delta z}{\overline{PQ}}.$$

As Q approaches P, these approach the direction cosines of the tangent at P. Hence

$$\cos \alpha = \lim_{Q \to P} \frac{\Delta x}{PQ} = \lim \frac{\Delta x}{\Delta s} \frac{\Delta s}{PQ}.$$

On the curve, x, y, z are functions of s. Hence

$$\lim \frac{\Delta x}{\Delta s} = \frac{dx}{ds}, \qquad \lim \frac{\Delta s}{PQ} = \lim \frac{\text{arc}}{\text{chord}} = 1.*$$

Therefore

$$\cos \alpha = \frac{dx}{ds}. \tag{125a}$$

Similarly,

$$\cos \beta = \frac{dy}{ds}, \qquad \cos \gamma = \frac{dz}{ds}. \tag{125a}$$

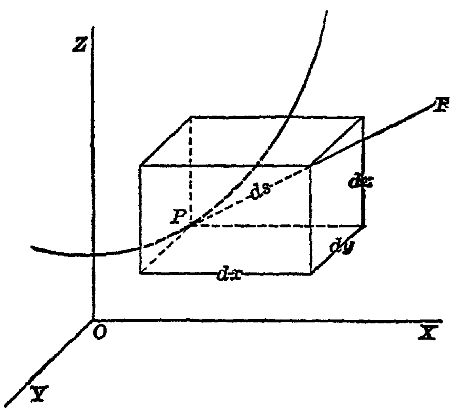

FIG. 125b.

These equations show that if a distance ds is measured along the tangent, dx, dy, dz are its projections on the coördinate axes (Fig. 125b). Since the square on the diagonal of a

* The proof that the limit of arc/chord is 1 was given in Art. 56 for the case of plane curves with continuous slope. A similar proof can be given for any curve, plane or space, that is continuous in direction.

rectangular parallelopiped is equal to the sum of the squares of its three edges,

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{125b}$$

Example. Find the direction cosines of the tangent to the parabola

$$x = at, \qquad y = bt, \qquad z = \tfrac{1}{4}\, ct^2$$

at the point where $t = 2$.

At $t = 2$ the differentials are

$$dx = a\, dt, \qquad dy = b\, dt, \qquad dz = \tfrac{1}{2}\, ct\, dt = c\, dt,$$
$$ds = \pm\sqrt{dx^2 + dy^2 + dz^2} = \pm\sqrt{a^2 + b^2 + c^2}\, dt.$$

There are two algebraic signs depending on the direction in which s is measured along the curve. If we take the positive sign, the direction cosines are

$$\frac{dx}{ds} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad \frac{dy}{ds} = \frac{b}{\sqrt{a^2 + b^2 + c^2}},$$
$$\frac{dz}{ds} = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

126. Equations of the Tangent Line. — It is shown in analytic geometry that the equations of a straight line through a point P_1 (x_1, y_1, z_1) with direction cosines proportional to A, B, C are

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}. \tag{126}$$

The direction cosines of the tangent line are proportional to dx, dy, dz. If then we replace A, B, C by numbers proportional to the values of dx, dy, dz at P_1, (126) will represent the tangent line at P_1.

Example. Find the equations of the tangent to the curve

$$x = t, \qquad y = t^2, \qquad z = t^3$$

at the point where $t = 1$.

The point of tangency is $t = 1$, $x_1 = 1$, $y_1 = 1$, $z_1 = 1$. At this point the differentials are

$$dx : dy : dz = dt : 2\,t\,dt : 3\,t^2\,dt = 1 : 2 : 3.$$

The equations of the tangent line are then

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}.$$

127. Tangent Plane to a Surface. — Let

$$F(x, y, z) = 0$$

be the equation of a surface. If the point $P(x, y, z)$ describes a curve on the surface, the differentials of its coordinates satisfy the equation

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz = 0.$$

In particular, the differentials at the point $P_1(x_1, y_1, z_1)$ satisfy the equation

$$\left(\frac{\partial F}{\partial x}\right)_1 dx + \left(\frac{\partial F}{\partial y}\right)_1 dy + \left(\frac{\partial F}{\partial z}\right)_1 dz = 0, \qquad \text{(127a)}$$

where $\left(\frac{\partial F}{\partial x}\right)_1, \left(\frac{\partial F}{\partial y}\right)_1, \left(\frac{\partial F}{\partial z}\right)_1$ are the values of the functions at $P_1(x_1, y_1, z_1)$.

If $P(x, y, z)$ is any point on the tangent line through P_1, the differentials dx, dy, dz at P_1 are proportional to $x - x_1$, $y - y_1$, $z - z_1$. Since the differentials in (127a) may be replaced by any proportional numbers,

$$\left(\frac{\partial F}{\partial x}\right)_1 (x - x_1) + \left(\frac{\partial F}{\partial y}\right)_1 (y - y_1) + \left(\frac{\partial F}{\partial z}\right)_1 (z - z_1) = 0. \quad \text{(127b)}$$

Being of the first degree in x, y, z this equation represents a plane. If different curves are drawn on the surface through

P_1 their tangent lines at P_1 all satisfy the equation and so all lie in the plane. This plane containing all the tangent lines at P_1 is called the tangent plane to the surface at P_1.

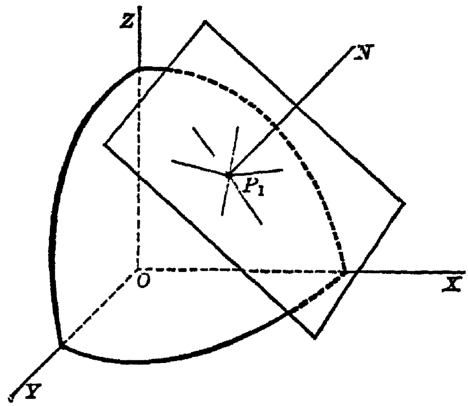

FIG. 127

128. Normal to a Surface. — Through P_1 draw a line P_1N with direction cosines proportional to $\left(\frac{\partial F}{\partial x}\right)_1$, $\left(\frac{\partial F}{\partial y}\right)_1$, $\left(\frac{\partial F}{\partial z}\right)_1$. Equation (127b) expresses that this line is perpendicular to any line P_1P in the tangent plane. It is therefore perpendicular to the tangent plane and is called the *normal* to the surface at P_1. Since the direction cosines of the normal are proportional to the partial derivatives of $F(x, y, z)$ at P_1 the equations of the normal are

$$\frac{x - x_1}{\left(\frac{\partial F}{\partial x}\right)_1} = \frac{y - y_1}{\left(\frac{\partial F}{\partial y}\right)_1} = \frac{z - z_1}{\left(\frac{\partial F}{\partial z}\right)_1}. \tag{128}$$

In applying equations (127b) and (128) the values of the partial derivatives may be replaced by any numbers having the same ratios.

Example. Find the equations of the normal line and tangent plane at the point $(1, -1, 2)$ of the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 3x + 12,$$

The equation given is equivalent to

$$x^2 + 2\,y^2 + 3\,z^2 - 3\,x - 12 = 0.$$

The direction cosines of its normal are proportional to the partial derivatives

$$2\,x - 3 : 4\,y : 6\,z.$$

At the point $(1, -1, 2)$, these are proportional to

$$A : B : C = -1 : -4 : 12 = 1 : 4 : -12.$$

The equations of the normal are

$$\frac{x-1}{1} = \frac{y+1}{4} = \frac{z-2}{-12}.$$

The equation of the tangent plane is

$$x - 1 + 4\,(y+1) - 12\,(z-2) = 0.$$

129. Maxima and Minima of Functions of Two or More Independent Variables. — A maximum value of a function is a value greater than any given by neighboring values of the variables. A minimum value is one less than any given by neighboring values of the variables.

In case of two independent variables, x, y, a function

$$z = F(x, y)$$

can be represented graphically by a surface. A maximum value occurs at the top of an elevation, a minimum at the bottom of a depression, in the surface. In either case the tangent plane is horizontal and so

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0. \qquad (129a)$$

If u is a function of any number of independent variables x, y, z, etc., it is clear that a maximum or minimum value of

u remains a maximum or minimum when only one variable changes. If then all the independent variables but x are kept constant (and $\frac{\partial u}{\partial x}$ is continuous), a maximum or minimum must satisfy the condition

$$\frac{\partial u}{\partial x} = 0. \tag{129b}$$

Therefore, *if the first partial derivatives of u with respect to the independent variables are continuous, those derivatives must be zero when u is a maximum or minimum.*

When the partial derivatives are all zero, the total differential is zero. Thus if u is a function of x, y, z and the partial derivatives are all zero,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \tag{129c}$$

for all values of dx, dy, dz.

To find the maximum and minimum values of a function we equate its partial derivatives with respect to the *independent* variables (or its total differential) to zero and solve the resulting equations. It is usually possible to decide from the problem whether a value thus found is a maximum, a minimum, or neither.

Example 1. Show that the maximum rectangular parallelepiped with a given area of surface is a cube.

Let x, y, z be the edges of the parallelepiped. If V is the volume and A the area of its surface

$$V = xyz, \qquad A = 2\,xy + 2\,xz + 2\,yz.$$

Two of the variables x, y, z are independent. Let them be x, y. Then

$$z = \frac{A - 2\,xy}{2\,(x + y)}.$$

Therefore

$$V = \frac{xy(A - 2\,xy)}{2\,(x + y)},$$

$$\frac{\partial V}{\partial x} = \frac{y^2}{2}\left[\frac{A - 2\,x^2 - 4\,xy}{(x + y)^2}\right] = 0,$$

$$\frac{\partial V}{\partial y} = \frac{x^2}{2}\left[\frac{A - 2\,y^2 - 4\,xy}{(x + y)^2}\right] = 0.$$

The values $x = 0$, $y = 0$ cannot give maxima. Hence

$$A - 2\,x^2 - 4\,xy = 0, \qquad A - 2\,y^2 - 4\,xy = 0.$$

Solving these equations simultaneously with

$$A = 2\,xy + 2\,xz + 2\,yz,$$

we get

$$x = y = z = \sqrt{\frac{A}{6}}.$$

We know there is a maximum. Since the equations give only one solution it must be the maximum.

Example 2. Find the point in the plane

$$x + 2\,y + 3\,z = 14$$

nearest to the origin.

The distance from any point (x, y, z) of the plane to the origin is

$$D = \sqrt{x^2 + y^2 + z^2}.$$

If this is a minimum

$$d \cdot D = \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2 + y^2 + z^2}} = 0,$$

that is,

$$x\,dx + y\,dy + z\,dz = 0. \tag{129d}$$

From the equation of the plane we get

$$dx + 2\,dy + 3\,dz = 0. \tag{129e}$$

The only equation connecting x, y, z is that of the plane. Consequently, dx, dy, dz can have any values satisfying this last equation. If x, y, z are so chosen that D is a minimum (129d) must be satisfied by all of these values. If two linear equations have the same solutions, one is a multiple of the other. Corresponding coefficients are proportional. The coefficients of dx, dy, dz in (129d) are x, y, z. Those in (129e) are 1, 2, 3. Hence

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}.$$

Solving these simultaneously with the equation of the plane, we get $x = 1$, $y = 2$, $z = 3$. There is a minimum. Since we get only one solution, it is the minimum.

EXERCISES

Find the equations of the tangent lines to the following curves at the points indicated:

1. $x = t$, $y = 2\,t$, $z = t^2$, at $t = 1$.

2. $x = \sec t$, $y = \tan t$, $z = t$, at $t = \frac{\pi}{4}$.

3. $x = e^t$, $y = e^{-t}$, $z = \sin t$, at $t = 0$.

4. Find the angle at which the helix

$$x = a\cos\theta, \qquad y = a\sin\theta, \qquad z = k\theta$$

cuts the generators of the cylinder $x^2 + y^2 = a^2$ on which it lies.

5. Find the angle at which the conical helix

$$x = t\cos t, \qquad y = t\sin t, \qquad z = t$$

cuts the generators of the cone $x^2 + y^2 = z^2$ on which it lies.

Find the equations of the normal and tangent plane to each of the following surfaces at the point indicated:

6. Sphere $x^2 + y^2 + z^2 = 9$, at (1, 2, 2).

7. Cylinder $x^2 - y^2 = 8$, at (3, −1, 1).

8. Cone $z^2 = x^2 + y^2$, at (3, 4, 5).

9. Paraboloid $x^2 + y^2 = 2\,z$, at (1, 3, 5).

10. Find the locus of points on the cylinder

$$(x + y)^2 + (y - z)^2 = 4$$

where the normal is parallel to the xy-plane.

11. An open rectangular box is to have a given capacity. Find the dimensions of the box requiring least material for its construction.

12. The vertices of a triangle are (x_1, y_1), (x_2, y_2), (x_3, y_3). Find the point in the plane of the triangle such that the sum of the squares of its distances from the vertices is least.

13. Find the points on the surface $z^2 = xy + 1$ at least distance from the origin.

14. A tent having the form of a cylinder surmounted by a cone is to contain a given volume. Find its dimensions if the canvas required is a minimum.

15. When an electric current of strength I flows through a wire of resistance R the heat produced is proportional to I^2R. Two terminals are connected by three wires of resistances R_1, R_2, R_3, respectively. A given current flowing between the terminals will divide between the wires in such a way that the total heat produced is a minimum. Show that the currents in the three wires will satisfy the equations

$$I_1R_1 = I_2R_2 = I_3R_3.$$

16. Show that the triangle of greatest area with a given perimeter is equilateral.

17. Two adjacent sides of a room are plane mirrors. A ray of light starting at a point P strikes one of the mirrors at Q, is reflected to a point R on the second mirror, and is there reflected to S. If P and S are in the same horizontal plane, find the positions of Q and R so that the path $PQRS$ shall be as short as possible.

CHAPTER XVII

DOUBLE INTEGRATION

130. Double Integrals. — The notation

$$\int_a^b \int_c^d f(x, y)\, dx\, dy$$

is used to represent the result of integrating first with respect to y (leaving x constant) between the limits c, d and then with respect to x between the limits a, b.

As here defined the first integration is with respect to the variable whose differential stands last and its limits are attached to the last integral sign. Some writers integrate in a different order. In reading an article it is therefore necessary to know what convention the author uses.

Example. Find the value of the double integral

$$\int_0^1 \int_{-x}^x (x^2 + y^2)\, dx\, dy.$$

We integrate first with respect to y between the limits $-x$, x, then with respect to x between the limits 0, 1. The result is

$$\int_0^1 \int_{-x}^x (x^2 + y^2)\, dx\, dy = \int_0^1 dx\, [x^2 y + \tfrac{1}{3} y^3]_{-x}^x = \int_0^1 \tfrac{8}{3} x^3\, dx = \tfrac{2}{3}.$$

131. Area as a Double Integral. — Divide the area between two curves $y = f(x)$, $y = F(x)$ into strips of width Δx. Let P be the point (x, y) and Q the point $(x + \Delta x, y + \Delta y)$. The area of the rectangle PQ is $\Delta x\, \Delta y$. The area of the rectangle RS (Fig. 131a) is

$$\Delta x \sum_{f(x)}^{F(x)} \Delta y = \Delta x \int_{f(x)}^{F(x)} dy.$$

The area bounded by the ordinates $x = a$, $x = b$ is then

$$A = \lim_{\Delta x \doteq 0} \sum_a^b \Delta x \int_{f(x)}^{F(x)} dy = \int_a^b \int_{f(x)}^{F(x)} dx\, dy.$$

If it is simpler to cut the area into strips parallel to the x-axis, the area is

$$A = \int \int dy\, dx,$$

the limits in the first integration being the values of x at the ends of a variable strip; those in the second integration, the values of y giving the limiting strips.

Example. Find the area bounded by the parabola $y^2 = 4\,ax + 4\,a^2$ and the straight line $y = 2\,a - x$ (Fig. 131b).

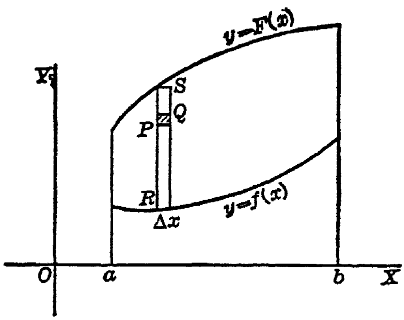

FIG. 131a.

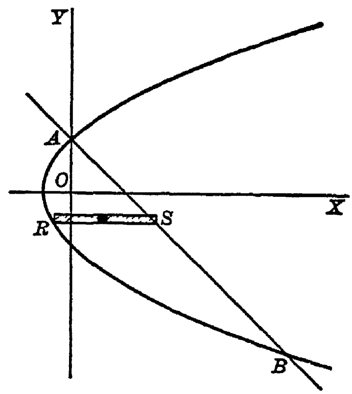

FIG. 131b.

Solving simultaneously, we find that the parabola and the line intersect at A $(0,\ 2\,a)$ and B $(8\,a,\ -6\,a)$. Draw the strips parallel to the x-axis. The area is

$$A = \int_{-6a}^{2a} \int_{\frac{y^2-4a^2}{4a}}^{2a-y} dy\, dx = \int_{-6a}^{2a} \left(2\,a - y - \frac{y^2 - 4\,a^2}{4\,a}\right) dy = \frac{64}{3}\,a^2.$$

The limits in the first integration are the values of x at R and S, the ends of the variable strip. The limits in the second integration are the values of y at B and A, corresponding to the outside strips.

132. Volume by Double Integration. — To find the volume under a surface $z = f(x, y)$ and over a given region in the xy-plane.

The volume of the prism PQ standing on the base $\Delta x\,\Delta y$ (Fig. 132a) is

$$z\,\Delta x\,\Delta y.$$

The volume of the plate RT is then

$$\lim_{\Delta y \doteq 0} \sum_R^S z\,\Delta x\,\Delta y = \Delta x \int_{f(x)}^{F(x)} z\,dy,$$

$f(x)$, $F(x)$ being the values of y at R, S. The entire volume is the limit of the sum of such plates

$$\lim_{\Delta x \doteq 0} \sum_a^b \Delta x \int_{f(x)}^{F(x)} z\,dy = \int_a^b \int_{f(x)}^{F(x)} z\,dx\,dy,$$

a, b being the values of x corresponding to the outside plates.

Example. Find the volume bounded by the surface $az = a^2 - x^2 - 4y^2$ and the xy-plane.

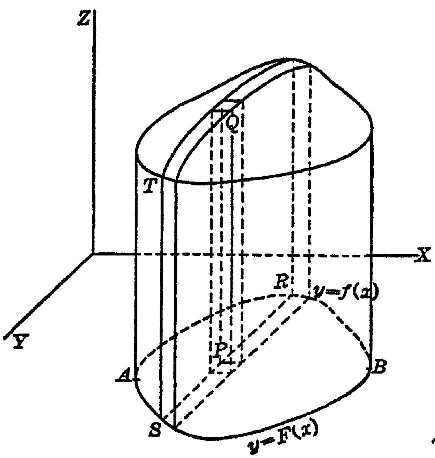

FIG. 132a.

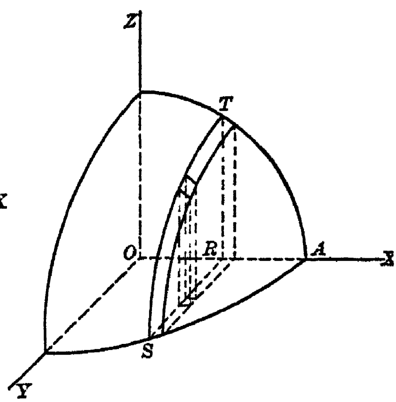

FIG. 132b.

Fig. 132b shows one-fourth of the required volume. At R, $y = 0$. At S, $z = 0$ and so

$$y = \tfrac{1}{2}\sqrt{a^2 - x^2}.$$

The limiting values of x at O and A are 0 and a. Therefore

$$v = 4\int_0^a \int_0^{\frac{1}{2}\sqrt{a^2-x^2}} z\,dx\,dy = 4\int_0^a \int_0^{\frac{1}{2}\sqrt{a^2-x^2}} \frac{1}{a}(a^2 - x^2 - 4\,y^2)\,dx\,dy$$
$$= \frac{4}{3\,a}\int_0^a (a^2 - x^2)^{\frac{3}{2}}\,dx = \frac{\pi a^3}{4}.$$

133. The Double Integral as the Limit of a Double Summation. — Divide a plane area by lines parallel to the coordinate axes into rectangles with sides Δx and Δy. Let (x, y) be any point within one of these rectangles. Form the product

$$f(x, y)\,\Delta x\,\Delta y.$$

This product is equal to the volume of the prism standing on the rectangle as base and reaching the surface $z = f(x, y)$ at some point over the base. Take the sum of such products

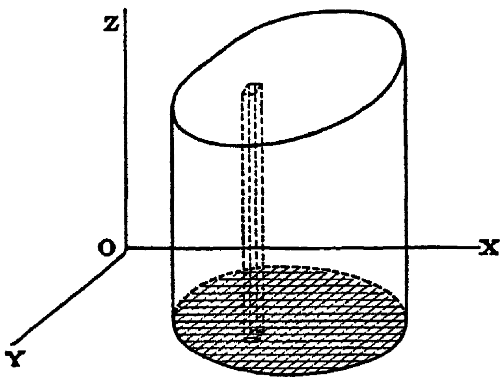

FIG. 133a.

for all the rectangles that lie entirely within the area. We represent this sum by the notation

$$\sum \sum f(x, y)\,\Delta x\,\Delta y.$$

When Δx and Δy are taken smaller and smaller, this sum approaches as limit the double integral

$$\int\int f(x, y)\,dx\,dy,$$

with the limits determined by the given area; for it approaches the volume over the area and that volume is equal to the double integral.

Whenever then a quantity is a limit of a sum of the form

$$\sum \sum f(x, y)\, \Delta x\, \Delta y$$

its value can be found by double integration. Furthermore, in the formation of this sum, infinitesimals of higher order than $\Delta x\, \Delta y$ can be neglected without changing the limit. For, if $\epsilon\, \Delta x\, \Delta y$ is such an infinitesimal, the sum of the errors thus made is

$$\sum \sum \epsilon\, \Delta x\, \Delta y.$$

When Δx and Δy approach zero, ϵ approaches zero. The sum of the errors approaches zero, since it is represented by a volume with finite base and altitude approaching zero.

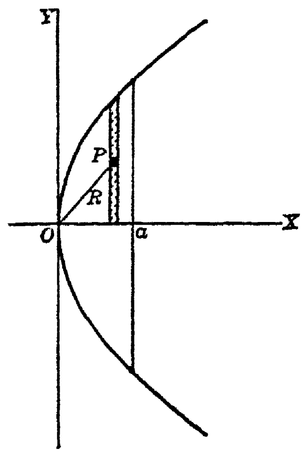

FIG. 133b.

Example 1. An area is bounded by the parabola $y^2 = 4\,ax$ and the line $x = a$. Find its moment of inertia about the axis perpendicular to its plane at the origin.

Divide the area into rectangles $\Delta x\, \Delta y$. The distance of any point $P\ (x, y)$ from the axis perpendicular to the plane at O is $R = OP = \sqrt{x^2 + y^2}$. If then (x, y) is a point within one of the rectangles, the moment of inertia of that rectangle is

$$R^2\, \Delta x\, \Delta y = (x^2 + y^2)\, \Delta x\, \Delta y,$$

approximately. That the result is approximate and not exact is due to the fact that different points in the rectangle differ slightly in distance from the axis. This difference is,

however, infinitesimal and, since R^2 is multiplied by $\Delta x\, \Delta y$, the resulting error is of higher order than $\Delta x\, \Delta y$. Hence in the limit

$$I = \int_0^a \int_{-2\sqrt{ax}}^{2\sqrt{ax}} (x^2 + y^2)\, dx\, dy = \frac{344}{105} a^4.$$

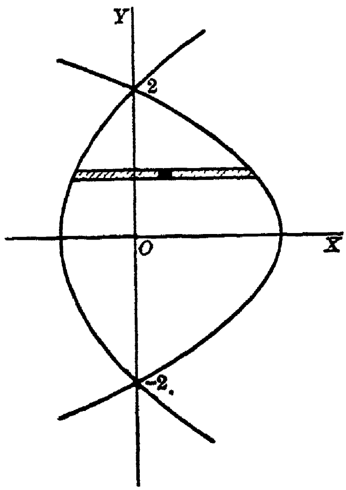

FIG. 133c.

Ex. 2. Find the center of gravity of the area bounded by the parabolas $y^2 = 4x + 4$, $y^2 = -2x + 4$.

By symmetry the center of gravity is seen to be on the x-axis. Its abscissa is

$$\bar{x} = \frac{\int x\, dA}{A}.$$

If we wish to use double integration we have merely to replace dA by $dx\, dy$ or $dy\, dx$. From the figure it is seen that the first integration should be with respect to x. Hence

$$\bar{x} = \frac{\int_{-2}^{2} \int_{\frac{1}{4}(y^2-4)}^{\frac{1}{2}(4-y^2)} x\, dy\, dx}{\int_{-2}^{2} \int_{\frac{1}{4}(y^2-4)}^{\frac{1}{2}(4-y^2)} dy\, dx} = \frac{\frac{16}{5}}{8} = \frac{2}{5}.$$

EXERCISES

Find the values of the following double integrals:

1. $\int_2^4 \int_1^2 xy\, dx\, dy.$
2. $\int_0^1 \int_0^y (x^2 + y^2)\, dy\, dx.$
3. $\int_1^2 \int_x^{2x} \frac{dx\, dy}{(x+y)^2}.$
4. $\int_0^\pi \int_0^{a \sin\theta} r\, d\theta\, dr.$
5. $\int_0^{2\pi} \int_0^\infty e^{-kr^2} r\, d\theta\, dr.$
6. $\int_0^a \int_0^{\sqrt{a^2-y^2}} dy\, dx.$

7. Find the area bounded by the parabola $y^2 = 3x$ and the line $x = y$.

8. Find the area bounded by the parabola $y^2 = 4ax$, the line $x + y = 3a$, and the x-axis.

9. Find the area enclosed by the ellipse

$$(y - x)^2 + x^2 = 1.$$

10. Find the volume under the paraboloid $z = x^2 + y^2$ and over the square $x = \pm 1$, $y = \pm 1$ in the xy-plane.

11. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$, and the plane $x + y + z = 2$

12. Find the volume in the first octant bounded by the paraboloid $z = xy$ and the plane $x + y = 1$.

13. Find the moment of inertia of a square of side $2a$ about the axis perpendicular to its plane at its center.

14. Find the moment of inertia of the triangle bounded by the lines $x + y = 2$, $x = 2$, $y = 2$, about the x-axis.

15. Find the moment of inertia of a cube of side a and mass M about an edge.

16. A wedge is cut from a cylinder of radius a by a plane tangent to the base and inclined 45° to the base. Find its moment of inertia about the axis of the cylinder.

17. Find the center of gravity of the area bounded by the coordinate axes and the lines $y = x + 1$, $x = 3$.

18. Find the center of gravity of the area bounded by the parabola $y^2 = 4ax + 4a^2$ and the line $y = 2a - x$.

19. Find the center of gravity of the pyramid bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + 2y + 3z = 6$.

20. Find the volume generated by rotating about the y-axis the area within the circle $x^2 + y^2 = 5a^2$ and the parabola $y^2 = 4ax$.

134. Double Integration. Polar Coördinates. — Pass through the origin a series of lines making with each other equal angles $\Delta\theta$. Construct a series of circles with centers at the origin and radii differing by Δr. The lines and circles divide the plane into curved quadrilaterals (Fig. 134a).

Let r, θ be the coördinates of P, $r + \Delta r$, $\theta + \Delta\theta$ those of Q. Since PR is the arc of a circle of radius r and subtends the angle $\Delta\theta$ at the center, $PR = r\,\Delta\theta$. Also $RQ = \Delta r$.

When Δr and $\Delta\theta$ are very small PRQ will be approximately a rectangle with area

$$PR \cdot RQ = r\,\Delta\theta\,\Delta r.$$

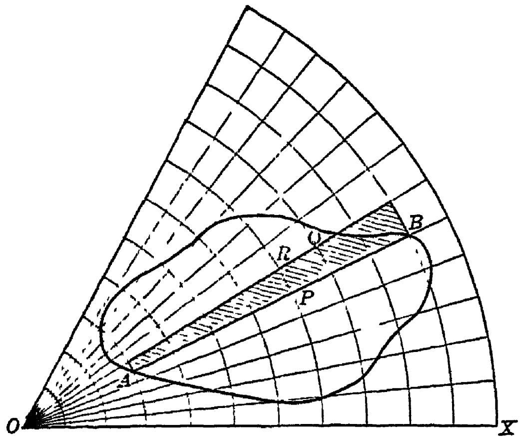

FIG. 134a.

It is very easy to show that the error is an infinitesimal of higher order than $\Delta\theta\,\Delta r$. (See Ex. 5, page 251.) Hence the sum

$$\sum\sum r\,\Delta\theta\,\Delta r,$$

taken for all the rectangles within a curve, gives in the limit the area of the curve in the form

$$A = \int\int r\,d\theta\,dr. \tag{134a}$$

The limits in the first integration are the values of r at the ends A, B of the strip across the area. The limits in the second integration are the values of θ giving the outside strips.

If it is more convenient the first integration may be with respect to θ. The area is then

$$A = \int\int r\,dr\,d\theta.$$

The first limits are the values of θ at the ends of a strip between two concentric circles (Fig. 134b). The second limits are the extreme values of r.

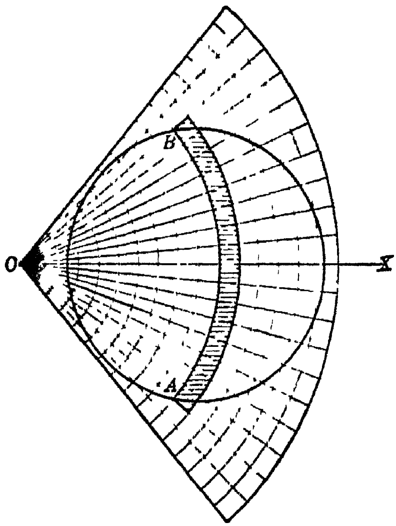

FIG. 134b.

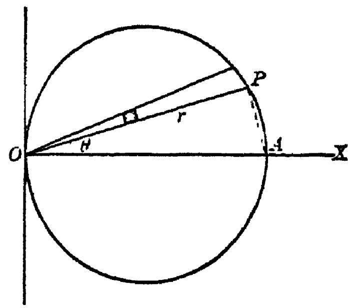

FIG. 134c.

The element of area in polar coordinates is

$$dA = r\,d\theta\,dr. \tag{134b}$$

We can use this in place of dA in finding moments of inertia, volumes, centers of gravity, or any other quantities expressed by integrals of the form

$$\int f(r, \theta)\,dA.$$

Example 1. Change the double integral

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2 + y^2)\,dx\,dy$$

to polar coördinates.

The integral is taken over the area of the semicircle $y = \sqrt{2ax - x^2}$ (Fig. 134c). In polar coördinates the equation of this circle is $r = 2a\cos\theta$. The element of area

$dx\,dy$ can be replaced by $r\,d\theta\,dr$.* Also $x^2 + y^2 = r^2$. Hence

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r^2\cdot r\,d\theta\,dr.$$

The limits for r are the ends of the sector OP. The limits for θ give the extreme sectors $\theta = 0$, $\theta = \frac{\pi}{2}$.

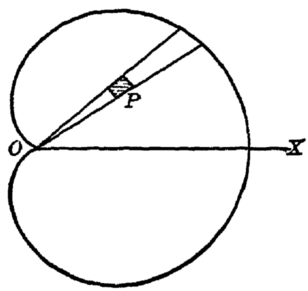

FIG. 134d.

Ex. 2. Find the moment of inertia of the area of the cardioid $r = a\ (1 + \cos\theta)$ about the axis perpendicular to its plane at the origin.

The distance from any point P (r, θ) (Fig. 134d) to the axis of rotation is

$$OP = r.$$

Hence the moment of inertia is

$$I = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cdot r\,d\theta\,dr = \frac{a^4}{2}\int_0^{\pi}(1+\cos\theta)^4\,d\theta = \frac{35}{16}\pi a^4.$$

Ex. 3. Find the center of gravity of the cardioid in the preceding problem.

The ordinate of the center of gravity is evidently zero. Its abscissa is

$$x = \frac{\int x\,dA}{\int dA} = \frac{2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\cos\theta\cdot r\,d\theta\,dr}{2\int\int r\,d\theta\,dr} = \frac{5}{6}a.$$

Ex. 4. Find the volume common to a sphere of radius $2\,a$ and a cylinder of radius a, the center of the sphere being on the surface of the cylinder.

* This does not mean that

$$dx\,dy = r\,d\theta\,dr,$$

but merely that the sum of all the rectangular elements in the circle is equal to the sum of all the polar elements.

Figure 134e shows one-fourth of the required volume. Take a system of polar coordinates in the xy-plane. On the element of area $r\,d\theta\,dr$ stands a prism of height

$$z = \sqrt{4\,a^2 - r^2}.$$

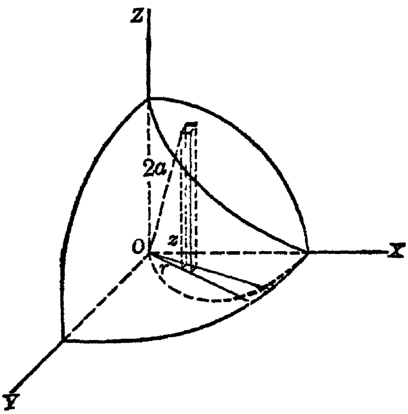

FIG. 134e.

The volume of the prism is $z \cdot r\,d\theta\,dr$ and the entire volume is

$$v = 4\int_0^{\frac{\pi}{2}}\int_0^{2\,a\cos\theta}\sqrt{4\,a^2-r^2}\cdot r\,d\theta\,dr = 4\int_0^{\frac{\pi}{2}}\left[\frac{(4\,a^2-r^2)^{\frac{3}{2}}}{-3}\right]_0^{2\,a\cos\theta} d\theta$$

$$= \frac{32\,a^3}{3}\int_0^{\frac{\pi}{2}}(1-\sin^3\theta)\,d\theta = \frac{16}{9}a^3(3\,\pi - 4).$$

EXERCISES

Find the values of the following integrals by changing to polar coordinates.

1. $\int_0^a\int_0^{\sqrt{a^2-y^2}}(x^2+y^2)\,dy\,dx.$

2. $\int_0^{2a}\int_0^{\sqrt{2\,ax-x^2}}dx\,dy.$

3. $\int_0^{\infty}\int_0^{\infty}\epsilon^{-(x^2+y^2)}\,dx\,dy.$

4. $\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dx\,dy.$

5. Find the area bounded by two circles of radii a, $a+\Delta a$ and two lines through the origin making with the initial line the angles α,

$\alpha + \Delta\alpha$, respectively. Show that when Δa and $\Delta\alpha$ approach zero the result differs from

$$a\,\Delta a\,\Delta\alpha$$

by an infinitesimal of higher order than $\Delta a\,\Delta\alpha$.

6. Find the areas of the two parts into which the circle $r = 1$ is cut by the spiral $r = \theta$.

7. A circular sector has a radius a and central angle $2\,\alpha$. Find the moment of inertia of its area about the bisector of the angle.

8. Find the center of gravity of the area in the preceding problem.

9. The center of a circle of radius $2\,a$ lies on a circle of radius a. Find the moment of inertia of the area between the two circles about the common tangent.

10. A hole of radius a is bored through a sphere of radius $2\,a$. If its axis passes through the center of the sphere, find the volume cut out.

11. Find the moment of inertia of a cone of radius a and mass M about its axis.

12. Find the moment of inertia of a sphere of radius a and mass M about a diameter.

13. A cylinder of radius a and mass M rotates about a generator. Find its moment of inertia.

14. Find the volume generated by rotating the cardioid $r = a(1 + \cos\theta)$ about the initial line.

15. Find the volume bounded by the xy-plane, the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 2\,ax$.

16. Find the volume bounded by the xy-plane, the paraboloid $az = x^2 + y^2$, and the cylinder $x^2 + y^2 = a^2$.

135. Area of a Surface. — Let an area A in one plane be projected upon another plane. The area of the projection is

$$A' = A\cos\phi,$$

where ϕ is the angle between the planes.

To show this divide A into rectangles by two sets of lines respectively parallel and perpendicular to the intersection MN of the two planes. Let a and b be the sides of one of these rectangles, a being parallel to MN. The projection of this rectangle will be a rectangle with sides

$$a' = a, \qquad b' = b\cos\phi,$$

and area

$$a'b' = ab \cos \phi.$$

The sum of the projections of all the rectangles is

$$\sum a'b' = \sum ab \cos \phi.$$

As the rectangles are taken smaller and smaller this approaches as limit

$$A' = A \cos \phi,$$

which was to be proved.

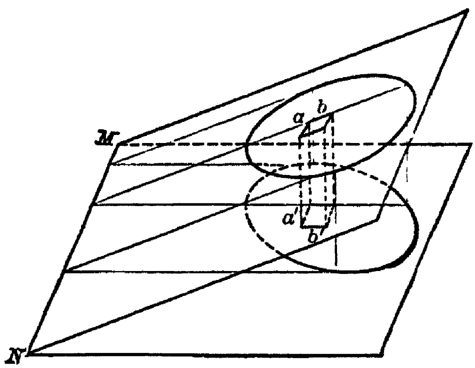

FIG 135a.

To find the area of a curved surface, resolve it into elements whose projections on a coördinate plane are equal to the differential of area dA in that plane. The element of surface can be considered as lying approximately in a tangent plane. Its area is, therefore, approximately

$$\frac{dA}{\cos \phi},$$

where ϕ is the angle between the tangent plane and the coördinate plane on which the area is projected. The area of the surface is the limit

$$S = \int \frac{dA}{\cos \phi}.$$

The angle between two planes is equal to that between the perpendiculars to the planes. Therefore ϕ is equal to the angle between the normal to the surface and the coordinate axis perpendicular to the plane on which we project.

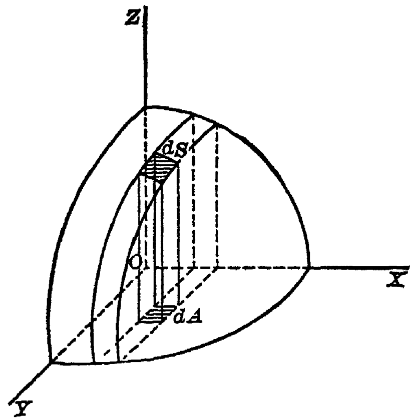

FIG. 135b.

If the equation of the surface is

$$F(x, y, z) = 0,$$

the direction cosines of its normal are proportional to $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ (Art. 128). Since the sum of the squares of the cosines is 1,

$$\cos\alpha = \frac{\frac{\partial F}{\partial x}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

and the values of $\cos\beta$ and $\cos\gamma$ are obtained by replacing $\frac{\partial F}{\partial x}$ by $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$. In finding areas the algebraic sign is assumed to be positive.

Example 1. Find the area of the sphere $x^2 + y^2 + z^2 = a^2$ within the cylinder $x^2 + y^2 = ax$.

Project on the xy-plane. The angle ϕ is then the angle γ between the normal to the sphere and the z-axis. Its cosine is

$$\cos\gamma = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{a}.$$

Using polar coördinates in the xy-plane,

$$z = \sqrt{a^2-x^2-y^2} = \sqrt{a^2-r^2}.$$

Hence the area of the surface is

$$S = \int \frac{dA}{\cos\gamma} = 4\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} \frac{ar\,d\theta\,dr}{\sqrt{a^2-r^2}} = 2a^2(\pi-2).$$

Ex. 2. Find the area of the surface of the cone $y^2+z^2=x^2$ in the first octant bounded by the plane $y+z=a$.

Project on the yz-plane. Then $\phi=\alpha$ and

$$\cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{2x^2}} = \frac{1}{\sqrt{2}}.$$

The area on the cone is therefore

$$S = \int_0^a\int_0^{a-y}\sqrt{2}\,dy\,dz = \frac{a^2\sqrt{2}}{2}.$$

EXERCISES

1. Find the area of the triangle cut from the plane

$$x+2y+3z=6$$

by the coördinate planes.

2. Find the area of the surface of the cylinder $x^2+y^2=4$ between the xy-plane and the plane $z=2x+4$.

3. Find the area of the surface of the cone $x^2+y^2=z^2$ cut out by the cylinder $x^2+y^2=2ax$.

4. Find the area in the plane $x+y+z=a$ bounded by the cylinder $x^2+y^2=a^2$.

5. Find the area of the surface $z^2=2xy$ above the xy-plane bounded by the planes $y=1$, $x=2$.

6. Find the area of the surface of the cylinder $x^2+y^2=2ax$ above the xy-plane bounded by the cone $x^2+y^2=z^2$.

7. Find the area of the surface of the paraboloid $y^2+z^2=2ax$ cut off by the plane $x=a$.

8. Find the area on the conical surface $x^2 + y^2 = z^2$ between the xy-plane and the plane $x + 2z = 3$.

9. Find the area cut from the plane by the cone in Ex. 8.

10. A hole of radius a is cut through a sphere of radius $2a$. If the axis of the hole is a diameter of the sphere, find the area of the surface cut out.

CHAPTER XVIII

TRIPLE INTEGRATION

136. Triple Integrals. — The notation

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\, dx\, dy\, dz$$

is used to represent the result of integrating first with respect to z (leaving x and y constant) between the limits z_1 and z_2, then with respect to y (leaving x constant) between the limits y_1 and y_2, and finally with respect to x between the limits x_1 and x_2.

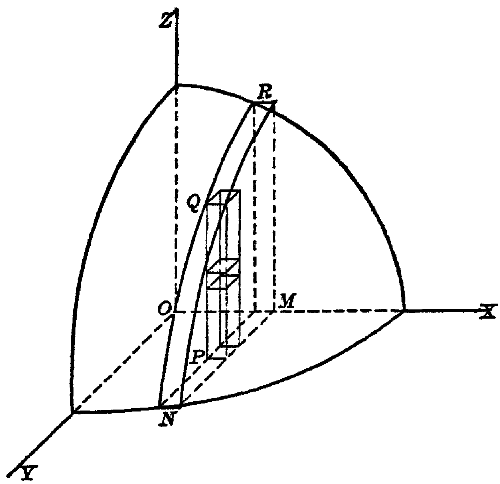

FIG. 136.

137. Rectangular Coördinates. — Divide a solid into rectangular parallelepipeds of volume $\Delta x\, \Delta y\, \Delta z$ by planes parallel to the coördinate planes. To find the volume of

the solid, first take the sum of the parallelepipeds in a vertical column PQ. The result is

$$\sum \Delta x \, \Delta y \, \Delta z = \Delta x \, \Delta y \int_{z_1}^{z_2} dz,$$

z_1 and z_2 being the values of z at the ends of the column. Then sum these columns along a base MN and so obtain the volume of the plate MNR. The result is

$$\lim_{\Delta y \doteq 0} \sum \Delta x \, \Delta y \int_{z_1}^{z_2} dz = \Delta x \int_{y_1}^{y_2} \int_{z_1}^{z_2} dy \, dz,$$

y_1 and y_2 being the limiting values of y in the plate. Finally, take the sum of these plates. The result is the triple integral

$$v = \lim_{\Delta x \doteq 0} \sum \Delta x \int_{y_1}^{y_2} \int_{z_1}^{z_2} dy \, dz = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} dx \, dy \, dz,$$

x_1, x_2 being the limiting values of x.

It may be more convenient to begin by integrating with respect to x or y. In any case the limits can be obtained from the consideration that the first integration is a summation of parallelepipeds to form a prism, the second a summation of prisms to form a plate, and the third integration a summation of plates.

Let (x, y, z) be any point of the parallelepiped $\Delta x \, \Delta y \, \Delta z$. Multiply $\Delta x \, \Delta y \, \Delta z$ by $f(x, y, z)$ and form the sum

$$\sum \sum \sum f(x, y, z) \, \Delta x \, \Delta y \, \Delta z$$

taken for all parallelepipeds in the solid. When Δx, Δy, and Δz approach zero, this sum approaches the triple integral

$$\int \int \int f(x, y, z) \, dx \, dy \, dz$$

as limit. It can be shown that terms of higher order than $\Delta x \, \Delta y \, \Delta z$ can be neglected in the sum without changing the limit.

The differential of volume in rectangular coördinates is

$$dv = dx\,dy\,dz.$$

This can be used in the formulas for moment of inertia, center of gravity, etc., those quantities being then determined by triple integration.

Example 1. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Fig. 136 shows one-eighth of the required volume. Therefore

$$v = 8 \int \int \int dx\,dy\,dz.$$

The limits in the first integration are the values $z = 0$ at P and $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ at Q. The limits in the second integration are the values of y at M and N. At M, $y = 0$ and at N, $z = 0$, whence $y = b\sqrt{1 - \frac{x^2}{a^2}}$. Finally, the limits for x are 0 and a. Therefore

$$v = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\,dy\,dz = \tfrac{4}{3}\pi abc.$$

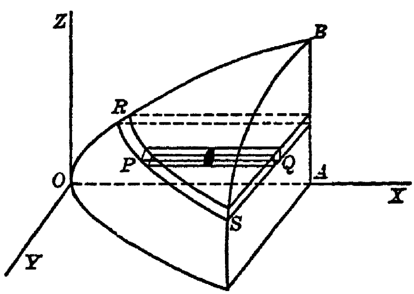

FIG. 137a.

Ex. 2. Find the center of gravity of the solid bounded by the paraboloid $y^2 + 2z^2 = 4x$ and the plane $x = 2$.

By symmetry $\bar{y}$ and $\bar{z}$ are zero. The x-coördinate is

$$\bar{x} = \frac{\int x\,dv}{\int dv} = \frac{4\int_0^2\int_0^{\sqrt{8-2z^2}}\int_{\frac{1}{4}(y^2+2z^2)}^2 x\,dz\,dy\,dx}{4\int\int\int dz\,dy\,dx} = \frac{4}{3}.$$

The limits for x are the values $x = \frac{1}{4}(y^2 + 2z^2)$ at P and $x = 2$ at Q. At S, $x = 2$ and $y = \sqrt{4x - 2z^2} = \sqrt{8 - 2z^2}$. The limits for y are, therefore, $y = 0$ at R and $y = \sqrt{8 - 2z^2}$ at S. The limits for z are $z = 0$ at A and $z = 2$ at B.

Ex. 3. Find the moment of inertia of a cube about an edge.

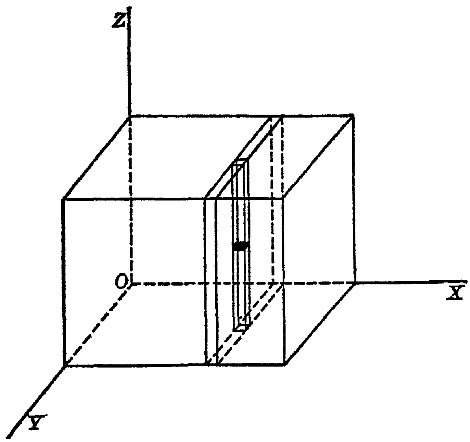

FIG. 137b.

Place the cube as shown in Fig. 137b and determine its moment of inertia about the z-axis. The distance of any point (x, y, z) from the z-axis is

$$R = \sqrt{x^2 + y^2}.$$

Hence the moment of inertia is

$$I = \int_0^a\int_0^a\int_0^a (x^2 + y^2)\,dx\,dy\,dz = \tfrac{2}{3}a^5,$$

where a is the edge of the cube.

EXERCISES

1. Find by triple integration the volume of the pyramid bounded by the coordinate planes and the plane $x + y + z = 1$.

2. Find the moment of inertia of the pyramid in Ex. 1 about the x-axis.

3. Express the volume of the cone

$$(z - 1)^2 = x^2 + y^2$$

in the first octant as a triple integral in 6 ways by integrating with dx, dy, dz arranged in all possible orders.

4. Find the volume bounded by the cylinder

$$z^2 = 1 - x - y$$

and the coordinate planes.

5. Find the volume bounded by the xy-plane and the surface

$$z = 1 - x^2 - y^2.$$

6. Find the volume bounded by the surfaces $y^2 = 4a^2 - 3ax$, $y^2 = ax$, $z = \pm h$.

7. Find the volume bounded by the plane $x = 0$ and the cylinders $y^2 + z^2 = a^2$, $z^2 = a^2 - ax$.

8. Find the volume cut from the paraboloid $z = x^2 + y^2$ by the plane $z = 2x$.

9. A wedge is cut from the base of a cylinder of radius a by a plane passing through a diameter AB of the base and inclined 45° to the base. Find its moment of inertia about the axis AB.

10. Find the center of gravity of a body having the form of an octant of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

11. Find the center of gravity of the solid bounded by the paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

and the plane $z = c$.

12. Find the moment of inertia of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

about the z-axis.

138. Cylindrical Coördinates. — Let M be the projection of P on the xy-plane. Let r, θ be the polar coördinates of M in the xy-plane. The cylindrical coördinates of P are r, θ, z.

From Fig. 138a it is evident that

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

By using these equations we can change any rectangular into a cylindrical equation.

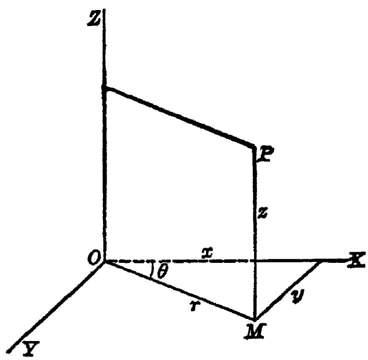

FIG. 138a.

The element of volume in cylindrical coördinates is the volume PQ, Fig. 138b, bounded by two cylindrical surfaces of radii r, $r + \Delta r$, two horizontal planes z, $z + \Delta z$, and two planes through the z-axis making angles θ, $\theta + \Delta\theta$ with OX. The base of PQ is equal to the polar element MN in the xy-plane. Its altitude PR is Δz. Hence

$$dv = r\,d\theta\,dr\,dz. \tag{138}$$

This value of dv can be used in the formulas for volume, center of gravity, moment of inertia, etc. In problems connected with cylinders, cones, and spheres, the resulting integrations are usually much easier in cylindrical than in rectangular coördinates.

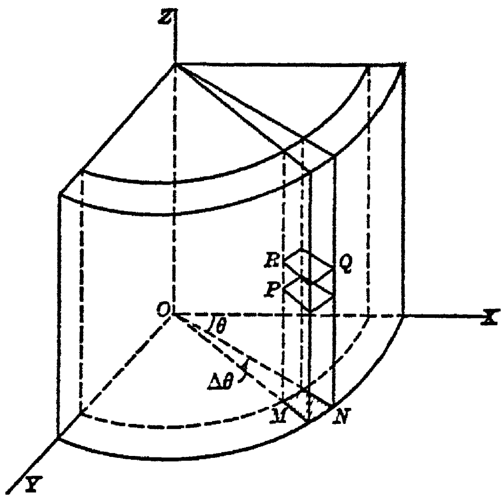

FIG. 138b.

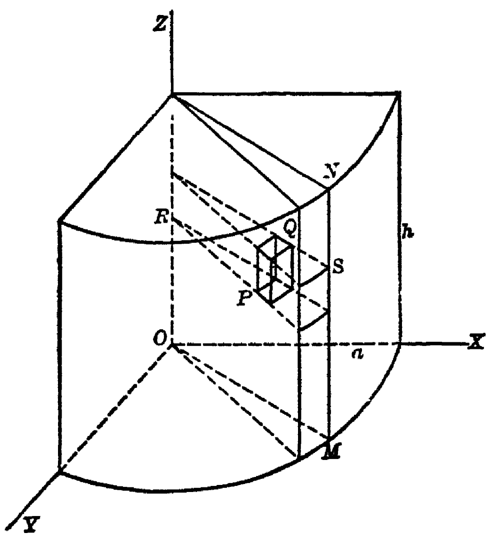

FIG. 138c.

Example 1. Find the moment of inertia of a cylinder about a diameter of its base.

Let the moment of inertia be taken about the x-axis, Fig. 138c. The square of the distance from the element PQ to the x-axis is

$$R^2 = y^2 + z^2 = r^2 \sin^2 \theta + z^2.$$

The moment of inertia is therefore

$$\int R^2\, dv = \int_0^{2\pi} \int_0^h \int_0^a (r^2 \sin^2 \theta + z^2)\, r\, d\theta\, dz\, dr$$
$$= \frac{\pi a^2 h}{12} (3\, a^2 + 4\, h^2).$$

The first integration is a summation for elements in the wedge RS, the second a summation for wedges in the slice OMN, the third a summation for all such slices.

Ex. 2. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = ax$, and the sphere $x^2 + y^2 + z^2 = a^2$.

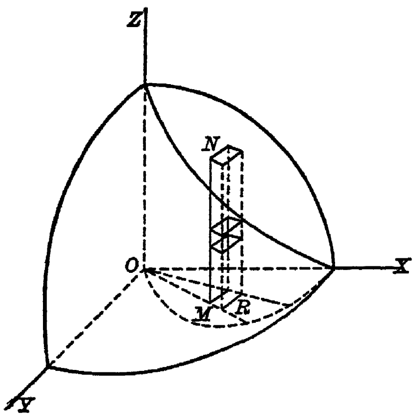

FIG. 138d.

In cylindrical coördinates, the equations of the cylinder and sphere are $r = a \cos \theta$ and $r^2 + z^2 = a^2$. The volume required is therefore

$$v = 2 \int_0^{\frac{\pi}{2}} \int_0^{a \cos \theta} \int_0^{\sqrt{a^2 - r^2}} r\, d\theta\, dr\, dz = \tfrac{1}{9}\, a^3 (3\pi - 4).$$

139. Spherical Coördinates. — The spherical coördinates of the point P (Fig. 139a) are $r = OP$ and the two angles θ and ϕ. From the diagram it is easily seen that

$$x = r \sin \phi \cos \theta,$$
$$y = r \sin \phi \sin \theta,$$
$$z = r \cos \phi.$$

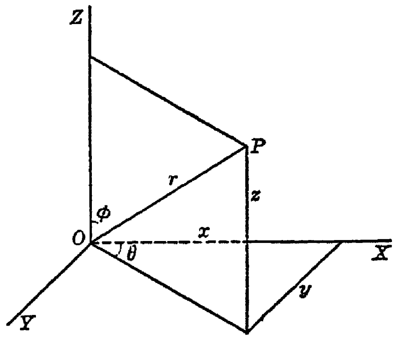

FIG. 139a.

The locus $r =$ const. is a sphere with center at O; $\theta =$ const. is the plane through OZ making the angle θ with OX; $\phi =$ const. is the cone generated by lines through O making the angle ϕ with OZ.

The element of volume is the volume $PQRS$ bounded

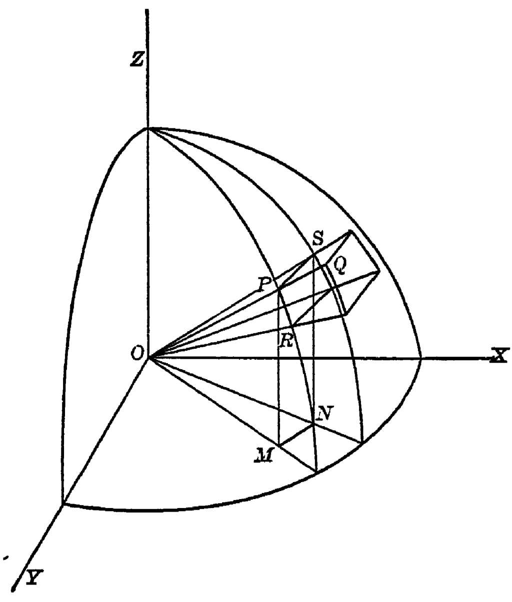

FIG. 139b.

by the spheres r, $r + \Delta r$, the planes θ, $\theta + \Delta\theta$, and the cones ϕ, $\phi + \Delta\phi$. When Δr, $\Delta\phi$, and $\Delta\theta$ are very small this is

approximately a rectangular parallelepiped. Since $OP = r$ and $POR = \Delta\phi$,

$$PR = r\,\Delta\phi.$$

Also $OM = OP \sin\phi$ and the arc PS is approximately equal to its projection MN, whence

$$PS = MN = r \sin\phi\,\Delta\theta,$$

approximately. Consequently

$$\Delta v = PR \cdot PS \cdot PQ = r^2 \sin\phi\,\Delta\theta \cdot \Delta\phi \cdot \Delta r,$$

approximately. When the increments are taken smaller and smaller, the result becomes more and more accurate. Therefore

$$dv = r^2 \sin\phi\, d\theta\, d\phi\, dr. \tag{139}$$

Spherical coördinates work best in problems connected with spheres. They are also very useful in problems where the distance from a fixed point plays an important role.

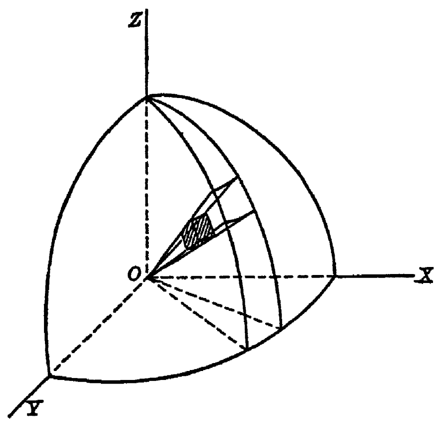

FIG. 139c.

Example. If the density of a solid hemisphere varies as the distance from the center, find its center of gravity.

Take the center of the sphere as origin and let the z-axis be perpendicular to the plane face of the hemisphere. By symmetry it is evident that $\bar{x}$ and $\bar{y}$ are zero. The density

is $\rho = kr$, where k is constant. Also $z = r\cos\phi$. Hence

$$\bar{z} = \frac{\int z\,dm}{\int dm} = \frac{\int krz\,dv}{\int kr\,dv}$$

$$= \frac{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a r^4 \cos\phi \sin\phi\,d\theta\,d\phi\,dr}{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a r^3 \sin\phi\,d\theta\,d\phi\,dr} = \frac{2}{5}a.$$

EXERCISES

1. Find the volume bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the paraboloid $x^2 + y^2 = 3z$.

2. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$.

3. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 2ax$, and the cone $z^2 = x^2 + y^2$.

4. Find the volume bounded by the surface $z = e^{-(x^2+y^2)}$ and the xy-plane.

5. Find the volume of the cylinder $x^2 + y^2 = 2ax$ intercepted between the paraboloid $x^2 + y^2 = 2az$ and the xy-plane.

6. Find the moment of inertia of a right cone, radius of base a, altitude h, and mass M, about an axis through the vertex parallel to the base.

7. Find the moment of inertia of a sphere of radius a and mass M about a tangent line.

8. Find the center of gravity of a right circular cone of density proportional to the distance from the vertex.

9. Find the center of gravity of the volume common to a cone of vertical angle 2α and a sphere of radius a, the vertex of the cone being the center of the sphere.

10. Find the center of gravity of the volume bounded by a spherical surface of radius a and two planes passing through the center including an angle of 60°.

11. The vertex of a cone of vertical angle 60° is on the surface of a sphere of radius a. If the axis of the cone is a diameter of the sphere, find the moment of inertia of the volume common to the cone and sphere about this axis.

12. A segment is cut from a sphere of radius a by a plane at distance $\frac{1}{2}a$ from the center. Find its moment of inertia about a diameter parallel to this plane.

13. The inner and outer radii of a spherical shell are $r = a$, $r = b$. If its density is ρ, find its moment of inertia about a diameter.

14. A spherical shell of inner radius 5 in. and outer radius 6 in. is cut by a plane at distance 3 in. from the center. Find the moment of inertia of the segment cut off about the diameter perpendicular to the plane.

15. A spherical shell of inner radius 5 in. and outer radius 6 in. is cut by two parallel planes on the same side of the center at distances 3 in. and 4 in. from the center. Find the moment of inertia of the segment between the planes about the diameter perpendicular to the planes.

16. Through a sphere of radius 5 in. a hole of radius 3 in. is bored, the axis of the hole being a diameter of the sphere. Find the moment of inertia of the ring thus formed about the axis of the hole.

17. The inner and outer radii of a hemispherical shell are $r = a$, $r = b$. Find its center of gravity.

18. An anchor ring of mass M is bounded by the surface generated by revolving a circle of radius a about an axis in its plane at distance b (greater than a) from its center. Find its moment of inertia about this axis.

CHAPTER XIX

SERIES AND APPROXIMATIONS

140. Series. — A sum

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

containing an infinite number of terms added in a definite order is called an infinite series. If the sum of the first n terms approaches a definite limit when n increases indefinitely the series is said to *converge* and the limit is called the sum of the series.

As an example, consider the geometrical series

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots.$$

This converges when r is numerically less than 1. For the sum of the first n terms is

$$S_n = a + ar + ar^2 + \cdots + a_{n-1}\, r^{n-1} = a\frac{1 - r^n}{1 - r}.$$

If r has a numerical value less than 1, r^n approaches zero and S_n approaches

$$S = \frac{a}{1 - r}$$

as n increases indefinitely.

If the sum of the first n terms does not approach a limit when n increases indefinitely, the series is called *divergent.* In such a case the sum may become infinite as in the series

$$1 + 2 + 4 + 8 + 16 + \cdots$$

or the sum while remaining finite may oscillate over a range of values as in the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \cdots,$$

where the sum of n terms is 1 or 0 according as n is odd or even.

141. Tests for Convergence. — Series are used to represent values of functions. An infinite series has a definite value only when it converges. Hence it is very important to know whether a given series converges or not.

The terms near the beginning of the series (if they are all finite) have no influence on the convergence or divergence of the series. This is determined by terms indefinitely far out in the series.

General Test. — For the series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

to converge it is necessary and sufficient that the sum of terms beyond u_n approach zero as n increases indefinitely.

For, if the series converges, the sum of n terms must approach a limit and so the sum of terms beyond the nth must approach zero.

Comparison Test. — A series is convergent if beyond a certain point its terms are in numerical value respectively less than those of a convergent series whose terms are all positive.

For, if a series converges, the sum of terms beyond the nth will approach zero as n increases indefinitely. If then another series has lesser corresponding terms, their sum will approach zero and the series will converge.

Ratio Test. — If the ratio $\frac{u_{n+1}}{u_n}$ of consecutive terms approaches a limit r as n increases indefinitely, the series

$$u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots$$

is convergent if r is numerically less than 1 and divergent if r is numerically greater than 1.

Since the limit is r, by taking n sufficiently large the ratio of consecutive terms can be made as nearly r as we please. If $r < 1$, let r_1 be a fixed number between r and 1. We can

take n so large that the ratio of consecutive terms is less than r_1. Then

$$u_{n+1} < r_1 u_n,\ u_{n+2} < r_1 u_{n+1} < r_1^2 u_n, \text{ etc.}$$

Beyond u_n the terms of the given series are therefore less than those of the geometrical progression

$$u_n + r_1 u_n + r_1^2 u_n + \cdots$$

which converges since r_1 is numerically less than 1. Consequently the given series converges.

If, however, r is greater than 1, the terms of the series must ultimately increase. The terms do not then approach zero and their sum cannot approach a limit.

Example. Find for what values of x the series

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

converges.

The ratio of consecutive terms is

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)x^{n+1}}{nx^n} = \left(1 + \frac{1}{n}\right)x.$$

The limit of this ratio is

$$r = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)x = x.$$

The series is then convergent when x is numerically less than 1.

142. Power Series. — A series of the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,$$

where $a_0, a_1, a_2, \cdots, a_n, \cdots$ are constants, is called a *power series.*

If a power series converges when $x = b$ it will converge for all values of x numerically less than b.

In fact, if the series converges when $x = b$ each term of

$$a_0 + a_1b + a_2b^2 + a_3b^3 + \cdots$$

will be less than a maximum value M, that is,

$$|a_nb^n| < M.*$$

Consequently

$$|a_n| < \frac{M}{|b|^n}.$$

The terms of the series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

are respectively less than those of the geometrical series

$$M + \frac{M}{|b|}|x| + \frac{M}{|b|^2}|x|^2 + \frac{M}{|b|^3}|x|^3 + \cdots,$$

in which the ratio of consecutive terms is

$$\frac{|x|}{|b|}.$$

If then $|x| < |b|$ the geometrical series and consequently the given series will converge.

If a power series diverges when $x = b$ it will diverge for all values of x numerically greater than b.

For it could not converge when $|x| > |b|$ since by the proof just given it would then converge at $x = b$.

In case of a power series there is then a number b such that the series converges when x is numerically less than b and diverges when x is numerically greater than b. At the limits, $x = \pm b$, the series may or may not converge. In more advanced treatises it is shown that within a common

* The notation $|a_nb^n|$ is used to represent the numerical value of a_nb^n without its algebraic sign. Thus

$$|-3| = |3| = 3.$$

region of convergence (but possibly not at the limits) power series can be added, subtracted, multiplied, divided, differentiated, and integrated like polynomials. In case of division by a series the region of convergence for the quotient will however usually not extend beyond any point at which the denominator is zero.

143. Maclaurin's Series. — Within its region of convergence a power series represents a continuous function. Suppose a given function $f(x)$ is expressible by means of a series of the form

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

the equality holding within the region of convergence of the series. In Art. 147 we shall show that most ordinary functions can be represented by series of this form.

To find the coefficients a_0, a_1, a_2, etc., we need merely substitute $x = 0$ in $f(x)$ and its derivatives. To illustrate the process suppose the function $f(x)$ is $\sin x$. Then

$$\sin x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots.$$

Differentiating both sides successively with respect to x, we get

$$\begin{aligned}
\cos x &= a_1 + 2\,a_2x + 3\,a_3x^2 + 4\,a_4x^3 + 5\,a_5x^4 + \cdots,\\
-\sin x &= 1\cdot 2\,a_2 + 2\cdot 3\,a_3x + 3\cdot 4\,a_4x^2 + 4\cdot 5\,a_5x^3 + \cdots,\\
-\cos x &= 1\cdot 2\cdot 3\,a_3 + 2\cdot 3\cdot 4\,a_4x + 3\cdot 4\cdot 5\,a_5x^2 + \cdots,\\
\sin x &= 1\cdot 2\cdot 3\cdot 4\,a_4 + 2\cdot 3\cdot 4\cdot 5\,a_5x + \cdots,\\
\cos x &= 1\cdot 2\cdot 3\cdot 4\cdot 5\,a_5 + \cdots,
\end{aligned}$$

Substituting $x = 0$ in each of these equations, we find

$$a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = -\frac{1}{3!},\ a_4 = 0,\ a_5 = \frac{1}{5!}.$$

Consequently

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Similarly, if

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

by repeated differentiation we find

$$f'(x) = a_1 + 2\,a_2x + 3\,a_3x^2 + \cdots,$$
$$f''(x) = 1\cdot 2\,a_2 + 2\cdot 3\,a_3x + \cdots,$$
$$f'''(x) = 1\cdot 2\cdot 3\,a_3 + \cdots,$$

where $f'(x)$, $f''(x)$, $f'''(x)$ represent the first, second, and third derivatives of $f(x)$.

By substituting $x = 0$ in these equations we get

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{1}{2!}\,f''(0), \quad a_3 = \frac{1}{3!}\,f'''(0),$$

where $f(0)$ is the value obtained by substituting $x = 0$ in $f(x)$. Hence

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \cdots. \qquad (143)$$

This is called *Maclaurin's series.*

Example 1. Expand $\ln(1 + x)$ in a Maclaurin series and determine its region of convergence.

In this case

$$f(x) = \ln(1 + x), \qquad f(0) = 0,$$
$$f'(x) = \frac{1}{1 + x}, \qquad f'(0) = 1,$$
$$f''(x) = -\frac{1}{(1 + x)^2}, \qquad f''(0) = -1,$$
$$f'''(x) = \frac{2}{(1 + x)^3}, \qquad f'''(0) = 2,$$
$$f^n(x) = \pm\frac{(n - 1)!}{(1 + x)^n}, \qquad f^n(0) = \pm(n - 1)!.$$

Hence

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n} + \cdots.$$

The ratio of the $(n + 1)$st to the nth term is

$$-\frac{n}{n+1}\,x,$$

which approaches the limit $-x$ when n increases indefinitely. Hence the series converges when x is numerically less than 1.

Example 2. Find the value of $\sin (15°)$ correct to four decimals.

The series for $\sin x$ obtained above is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots.$$

The circular measure of 15° is

$$x = \frac{15\,\pi}{180} = .26180.$$

A rough calculation shows that the third term in the series is too small to influence the fourth decimal. Hence

$$\begin{aligned}\sin (15°) &= .26180 - \frac{(.26180)^2}{6}\\ &= .26180 - .00299 = .2588.\end{aligned}$$

EXERCISES

Expand each of the following functions into a Maclaurin series and determine the region of its convergence:

1. $\operatorname{Cos} x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$

2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$

3. $2^x = 1 + x \ln 2 + \frac{(x \ln 2)^2}{2!} + \frac{(x \ln 2)^3}{3!} + \cdots.$

4. $(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}\, a^{n-2}x^2 + \cdots.$

5. Expand $\ln(1 + x)$ and $\ln(1 - x)$ and by combining the resulting series show that

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

6. Expand $e^x \cos x$ into a Maclaurin series and verify by multiplying the series for e^x by that for $\cos x$.

7. Expand $\tan x$ into a Maclaurin series and verify by dividing the series for $\sin x$ by that for $\cos x$.

8. By expanding $\cos 2x$ show that

$$\sin^2 x = \frac{1 - \cos 2x}{2} = 2\frac{x^2}{2!} - 2^3\frac{x^4}{4!} + 2^5\frac{x^6}{6!} \cdots .$$

9. If $|x| < 1$, show that

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots .$$

By integrating term by term show that

$$\tan^{-1} x = \int_0^x \frac{dx}{1 + x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

10. By a method similar to that used in Ex. 9, show that if $|x| < 1$

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\frac{x^7}{7} + \cdots .$$

11. By using the series of Ex. 10 and the equation

$$\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right)$$

calculate π to four decimals.

12. If $i = \sqrt{-1}$, by expanding into Maclaurin's series show that

$$e^{i\theta} = \cos\theta + i \sin\theta,$$
$$e^{-i\theta} = \cos\theta - i \sin\theta.$$

Determine the values of the following functions correct to four decimals:

13. $\cos 20°$.
14. $\tan 10°$.
15. $\sec 18°$.
16. $\sin 36°$.
17. $\sqrt{e}$.
18. $\ln (1 \cdot 2)$.

144. Taylor's Series. — When x is small the values of x^2, x^3, etc., diminish rapidly as the exponent increases. A satisfactory approximation for a given function can then usually be obtained by taking only a few terms in its Maclaurin's expansion. But if x is large it may be necessary to go far out in the series to reach a point where the terms are small enough to neglect. In such cases a different form of expansion called Taylor's series may be more convenient.

Let a be a constant and assume that $f(x)$ can be expressed as a series in powers of $x - a$. That is, let

$$f(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + a_3(x - a)^3 + \cdots.$$

By successive differentiation we get

$$\begin{aligned} f'(x) &= a_1 + 2\,a_2(x - a) + 3\,a_3(x - a)^2 + \cdots, \\ f''(x) &= 1 \cdot 2\,a_2 + 2 \cdot 3\,a_3(x - a) + \cdots, \\ f'''(x) &= 1 \cdot 2 \cdot 3\,a_3 + \cdots. \end{aligned}$$

Now let $x = a$. In each equation all the terms vanish except the first and so we find

$$a_0 = f(a), \quad a_1 = f'(a), \quad a_2 = \frac{f''(a)}{2!}, \quad \text{etc.}$$

Substituting these values, we get

$$f(x) = f(a) + f'(a)\,(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots. \tag{144}$$

This is called *Taylor's series.* To compute $f(x)$ by this formula we must know the values of $f(a)$, $f'(a)$, $f''(a)$, etc. *We must then assign a value to a such that $f(a)$, $f'(a)$, etc., are all known. Furthermore a should be chosen as close as possible to the value x at which $f(x)$ is wanted.* For, the smaller $x - a$, the fewer terms $(x - a)^2$, $(x - a)^3$, etc., need be computed to obtain a desired approximation.

Since Taylor's series converges most rapidly for values of x near a it is often called the expansion of the function in the neighborhood of $x = a$. Comparison of (143) and (144) shows that Maclaurin's series is merely the special form assumed by Taylor's series when $x = 0$. That is, Maclaurin's series gives the expansion of the function in the neighborhood of $x = 0$.

Example 1. Expand $\ln x$ in the neighborhood of $x = 1$. In this case $a = 1$,

$$\begin{aligned}
f(x) &= \ln x, & f(a) &= 0,\\
f'(x) &= \frac{1}{x}, & f'(a) &= 1,\\
f''(x) &= -\frac{1}{x^2}, & f''(a) &= -1,\\
f'''(x) &= \frac{2}{x^3}, & f'''(a) &= 2,\\
f''''(x) &= -\frac{6}{x^4}, & f''''(a) &= -6.
\end{aligned}$$

Therefore Taylor's expansion has the form

$$\ln x = (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots.$$

Example 2. Find $\tan 46°$ to four decimals.

The value nearest $46°$ for which $\tan x$ and its derivatives are known is $45°$. Therefore we let $x = \frac{\pi}{4}$

$$\begin{aligned}
f(x) &= \tan x, & f\left(\frac{\pi}{4}\right) &= 1,\\
f'(x) &= \sec^2 x, & f'\left(\frac{\pi}{4}\right) &= 2,\\
f''(x) &= 2\sec^2 x \tan x, & f''\left(\frac{\pi}{4}\right) &= 4,\\
f'''(x) &= 2\sec^4 x + 4\sec^2 x \tan^2 x, & f'''\left(\frac{\pi}{4}\right) &= 16.
\end{aligned}$$

Using these values Taylor's series for $\tan x$ assumes the form

$$\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \cdots.$$

When $x = 46°$,

$$x - \frac{\pi}{4} = \frac{\pi}{180} = .01745.$$

Powers beyond the second are evidently too small to influence the fourth decimal. Hence

$$\tan 46° = 1 + 2\,(.01745) + 2\,(.01745)^2$$
$$= 1 + .03490 + .00061 = 1.0355.$$

145. Remainder in Taylor's Series. — We have assumed that a function $f(x)$ can be represented by a convergent series in powers of $x - a$. We shall now show that under very general conditions this assumption is justified. For that purpose let R be the number such that

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2}\,(x-a)^2 + \cdots$$
$$+ \frac{f^n(a)}{n!}\,(x-a)^n + R.$$

This number R is called the remainder in Taylor's series. It is the error committed when we drop all powers beyond the nth. If R approaches the limit zero when n is indefinitely increased, Taylor's series is convergent.

We shall first find an expression for R in the special case when $n = 0$ and then extend the result to the general case.

146. Mean Value Theorem. — *If $f(x)$ and $f'(x)$ are continuous from $x = a$ to $x = b$ there is a value x_1 between a and b such that*

$$\frac{f(b) - f(a)}{b - a} = f'(x_1). \qquad (146a)$$

This is known as the mean value theorem. To prove it consider the curve $y = f(x)$. Since $f(a)$ and $f(b)$ are the ordinates at $x = a$ and $x = b$

$$\frac{f(b) - f(a)}{b - a} = \text{slope of chord } AB.$$

On the arc AB let P_1 be a point at maximum distance from the chord. The tangent at P_1 will be parallel to the chord

and so its slope $f'(x_1)$ will equal that of the chord. Hence

$$\frac{f(b) - f(a)}{b - a} = f'(x_1),$$

which was to be proved.

In particular, if

$$f(a) = f(b) = 0,$$

equation (146a) shows that $f'(x_1) = 0$. That is, *if $f(x)$ and $f'(x)$ are continuous, there is at least one real root of $f'(x) = 0$*

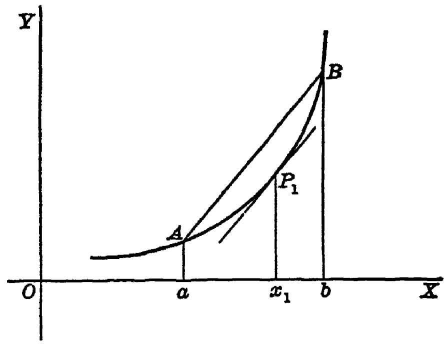

FIG. 146.

between each pair of real roots of $f(x) = 0$. This is known as Rolle's theorem.

Replacing b by x and solving for $f(x)$, equation (146a) becomes

$$f(x) = f(a) + (x - a)\, f'(x_1), \qquad (146b)$$

where x_1 is some value between a and x. This is a special case of a more general theorem which we shall now prove.

147. Taylor's Theorem. — *If $f(x)$ and all its derivatives used are continuous from a to x, there is a value x_1 between a and x such that*

$$f(x) = f(a) + f'(a)\,(x - a) + \frac{f''(a)}{2!}\,(x - a)^2 + \cdots$$
$$+ \frac{f^n(a)}{n!}\,(x - a)^n + \frac{f^{n+1}(x_1)}{(n+1)!}\,(x - a)^{n+1}.$$

To prove this let b be a value such that $f(x)$ and all the derivatives used are continuous between $x = a$ and $x = b$. Also let B be the number such that

$$f(b) = f(a) + f'(a)(b - a) + \cdots + \frac{f^n(a)}{n!}(b - a)^n + \frac{B}{(n+1)!}(b - a)^{n+1},$$

and let

$$\phi(x) = f(x) - f(a) - f'(a)(x - a) - \cdots - \frac{f^n(a)}{n!}(x - a)^n - \frac{B}{(n+1)!}(x - a)^{n+1}.$$

Now $\phi(x)$ is zero when $x = a$ and also when $x = b$. By Rolle's theorem there is then a value b_1 between a and b such that

$$\phi'(b_1) = 0.$$

Therefore

$$\phi'(x) = f'(x) - f'(a) - \cdots - \frac{f^n(a)}{(n-1)!}(x - a)^n - \frac{B}{n!}(x - a)^n$$

is zero when $x = a$ and also when $x = b_1$. There is consequently a value b_2 between $x = a$ and $x = b_1$ (and so between $x = a$ and $x = b$) such that

$$\frac{d}{dx}\phi'(x) = \phi''(x) = 0.$$

Continuing this process we ultimately reach a value b_{n+1} between $x = a$ and $x = b$ such that

$$\phi^{n+1}(b_{n+1}) = f^{n+1}(b_{n+1}) - B = 0.$$

If then $b_{n+1} = x_1$, we have

$$B = f^{n+1}(x_1)$$

and so

$$f(b) = f(a) + f'(a)(b - a) + \cdots + \frac{f^n(a)}{n!}(b - a)^n + \frac{f^{n+1}(x_1)}{(n+1)!}(b - a)^{n+1}.$$

Replacing b by x in this last equation we have Taylor's theorem.

The remainder in Taylor's series (defined in Art. 145) is then

$$R = \frac{f^{n+1}(x_1)}{(n+1)!}(x - a)^{n+1},$$

where x_1 is some value between a and x. If the limit of this is zero when n becomes infinite, the series converges and has a value equal to the function. By calculating this remainder we can justify any of the expansions we have used.

Placing $a = 0$ we obtain the remainder for Maclaurin's series in the form

$$R = \frac{f^{n+1}(x_1)}{(n+1)!}x^{n+1},$$

where x_1 is some number between 0 and x.

Example. Calculate ln (1.3) determining the approximate error.

Expanding in the neighborhood of $x = 1$, by Taylor's theorem we have

$$\ln x = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4x_1^4},$$

where x_1 lies between 1 and x. When $x = 1.3$ the value given by the first three terms is

$$\ln(1.3) = (.3) - \frac{(.3)^2}{2} + \frac{(.3)^3}{3} = .264,$$

the error being

$$R = -\frac{(x-1)^4}{4\,x_1^4}.$$

Since x_1 lies between 1 and 1.3 this will not be decreased in numerical value if we replace x_1 by 1. Hence

$$|R| \le \frac{(.3)^4}{4} \le .002.$$

The correct value obtained from a table is

$$\ln(1.3) = .2623$$

showing that the actual error in the above approximation is

$$|R| = .0017.$$

EXERCISES

Expand each of the following functions into Taylor's series using the given value of a:

1. $\sin x$, $\quad a = \frac{\pi}{6}$.
2. $\cos x$, $\quad a = \frac{\pi}{3}$.
3. $\sqrt{9 + x^2}$, $\quad a = 4$.
4. $\tan^{-1} x$, $\quad a = 1$.
5. Compute sin 65° to four decimal places by Taylor's series.
6. Compute cos 75° to four decimal places by Taylor's series.
7. Compute tan 55° to four decimal places by Taylor's series.
8. Given $\ln 5 = 1.6094$ find the value of $\ln 6$.
9. In computing $\sin x$ or $\cos x$ by Maclaurin's series show that the error resulting from stopping at any point is less than the next non-vanishing term in the series.
10. Find the value of sec (50°) to four decimals, proving the correctness of the approximation by finding an upper limit for the remainder.
11. Determine the value of $\sqrt[3]{26}$ to four decimals by expanding $\sqrt[3]{x}$ in a Taylor's series with $a = 25$ and finding the approximate error.
12. How many terms in the Maclaurin expansion for $\ln(1 + x)$ are needed to calculate $\ln(.8)$ to 6 decimal places?

148. Approximate Integration. — It is sometimes impossible to evaluate a definite integral

$$\int_a^b f(x)\,dx$$

by direct integration. In other cases an expression for the integral can be found but it does not have a form convenient for computation. In such cases the value can be found with any desired accuracy by methods of approximation.

The simplest procedure is to resolve the area represented by the integral into rectangles with equal bases Δx as was done in Chapter XII. If $y_1, y_2, \cdots, y_n$ are the ordinates

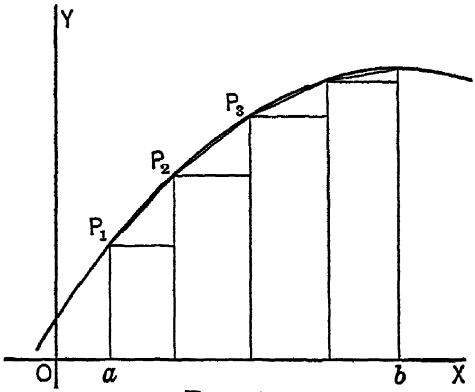

FIG. 148.

of the curve $y = f(x)$ at $x = a$, $x = b$, and the points of division, the integral is approximately equal to

$$\sum_a^b f(x)\ \Delta x = [y_1 + y_2 + \cdots + y_{n-1}]\ \Delta x.$$

A better approximation will, however, nearly always be obtained if we replace the arcs P_1P_2, P_2P_3, etc. (Fig. 148), by chords and in place of the rectangle in each section take the trapezoid between the x-axis and the chord at its top. The value thus obtained is

$$\int_a^b f(x)\,dx = \tfrac{1}{2}[y_1 + y_2]\ \Delta x + \tfrac{1}{2}[y_2 + y_3]\ \Delta x + \cdots$$
$$+ \tfrac{1}{2}[y_{n-1} + y^n]\ \Delta x$$
$$= \tfrac{1}{2}[y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n]\ \Delta x.$$

This expression for the integral is known as the *trapezoidal rule.* By taking Δx sufficiently small any desired approximation can be obtained.

Example. Find an approximate value for the integral

$$\int_0^1 \frac{dx}{1+x^3}.$$

Divide the interval from 0 to 1 into five parts

$$\Delta x = 0.2.$$

The values of the function

$$y = \frac{1}{1+x^3},$$

at $x = 0$, $x = 1$, and the points of division are

$$y_1 = 1.0000, \quad y_2 = .9921, \quad y_3 = .9399,$$
$$y_4 = .8224, \quad y_5 = .6614, \quad y_6 = .5000.$$

The trapezoidal rule gives for the integral

$$\int_0^1 \frac{dx}{1+x^3} = \tfrac{1}{2}\left[y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6\right](0.2) = .833.$$

The value found by integration is

$$\tfrac{1}{3}\ln 2 + \frac{\pi}{3\sqrt{3}} = .8356.$$

149. The Prismoidal Formula. — Let y_1, y_3, be two ordinates of a curve at distance h apart, and let y_2 be the ordinate midway between them. The area bounded by the x-axis, the curve, and the two ordinates is given approximately by the formula

$$A = \tfrac{1}{6}h(y_1 + 4y_2 + y_3). \tag{149a}$$

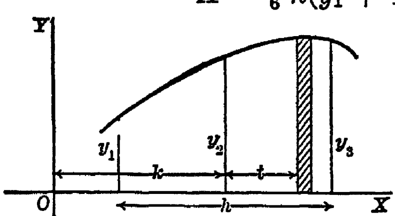

FIG. 149a.

This is called the prismoidal formula because of its similarity to the formula for the volume of a prismoid.

If the equation of the curve is

$$y = a + bx + cx^2 + dx^3, \tag{149b}$$

where a, b, c, d are constants (some of which may be zero), the prismoidal formula gives the exact area. To prove this

let k be the abscissa of the middle ordinate and t the distance of any other ordinate from it (Fig. 149a). Then

$$x = k + t.$$

If we substitute this value for x, (149b) takes the form

$$y = A + Bt + Ct^2 + Dt^3,$$

where A, B, C, D are constants. The ordinates y_1, y_2, y_3 are obtained by substituting $t = -\frac{h}{2}, 0, \frac{h}{2}$. Hence

$$y_1 + 4\,y_2 + y_3 = 6\,A + \tfrac{1}{2}\,Ch^2.$$

Also the area is

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} y\,dt = Ah + C\frac{h^3}{12}.$$

This is equivalent to

$$\frac{h}{6}\left(6\,A + \frac{1}{2}Ch^2\right) = \frac{h}{6}(y_1 + 4\,y_2 + y_3),$$

which was to be proved.

If the equation of the curve does not have the form (149b), it may be approximately equivalent to one of that type and so the prismoidal formula may give an approximate value for the area.

Example 1. Find the area bounded by the x-axis, the curve $y = e^{-x^2}$, and the ordinates $x = 0$, $x = 2$.

The integral

$$\int e^{-x^2}\,dx$$

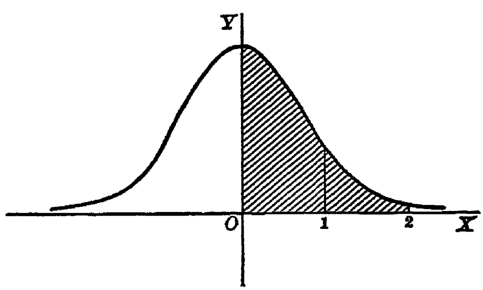

FIG. 149b.

cannot be expressed in terms of elementary functions. Therefore we cannot obtain the area by the methods that we

have previously used. The ordinates y_1, y_2, y_3, in this case are

$$y_1 = 1, \qquad y_2 = e^{-1}, \qquad y_3 = e^{-4}.$$

The prismoidal formula, therefore, gives

$$A = \frac{2}{6}\left(1 + \frac{4}{e} + \frac{1}{e^4}\right) = 0.830.$$

The answer correct to 3 decimals (obtained from a table) is 0.882.

Ex. 2. Find the length of the parabola $y^2 = 4x$ from $x = 1$ to $x = 5$.

The length is given by the formula

$$s = \int_1^5 \sqrt{\frac{x+1}{x}}\,dx.$$

By integration we find $s = 4.726$. To apply the prismoidal formula, let

$$y = \sqrt{\frac{x+1}{x}}.$$

Then $h = 4$,

$$y_1 = \sqrt{2}, \qquad y_2 = \sqrt{\tfrac{4}{3}}, \qquad y_3 = \sqrt{\tfrac{6}{5}},$$

and

$$s = \tfrac{4}{6}\left(\sqrt{2} + 4\sqrt{\tfrac{4}{3}} + \sqrt{\tfrac{6}{5}}\right) = 4.752.$$

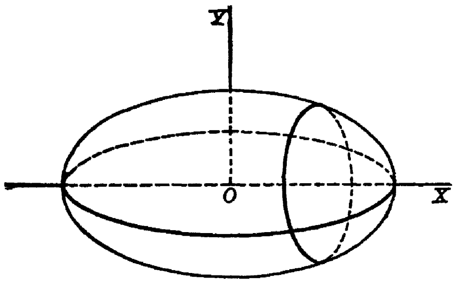

FIG. 149c.

Ex. 3. Find the volume of the spheroid generated by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis.

The section of the spheroid perpendicular to OX has the area

$$A = \pi y^2 = \pi b^2\left(1 - \frac{x^2}{a^2}\right).$$

Its volume is

$$V = \int_{-a}^{a} A\,dx.$$

Since A is a polynomial of the second degree in x (a special case of a third degree polynomial), the prismoidal formula gives the exact volume. The three cross-sections corresponding to $x = -a$, $x = 0$, $x = a$, are

$$A_1 = 0, \qquad A_2 = \pi b^2, \qquad A_3 = 0.$$

Hence

$$V = \frac{2a}{6}[A_1 + 4A_2 + A_3] = \frac{4}{3}\pi ab^2.$$

150. Simpson's Rule. — Divide the area between a curve and the x-axis into any even number of parts by means of equidistant ordinates $y_1, y_2, y_3, \ldots, y_n$. (An odd number of ordinates will be needed.) Simpson's rule for determining approximately the area between y_1 and y_n is

$$A = h\left(\frac{y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \cdots + y_n}{1 + 4 + 2 + 4 + 2 + \cdots + 1}\right), \quad (150)$$

h being the distance between the ordinates y_1 and y_n. In the numerator the end coefficients are 1. The others are alternately 4 and 2. The denominator is the sum of the coefficients in the numerator.

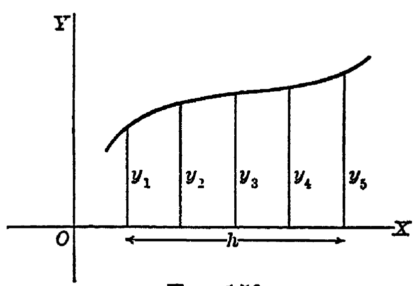

FIG. 150.

This formula is obtained by applying the prismoidal formula to the strips taken two at a time and adding the results. Thus if the area is divided into four strips by the ordinates y_1, y_2, y_3, y_4, y_5, the part between y_1 and y_3 has a base equal to $\frac{h}{2}$. Its area as given by the prismoidal formula is

$$\frac{1}{6}\frac{h}{2}(y_1 + 4y_2 + y_3).$$

Similarly the area between y_3 and y_5 is

$$\frac{1}{6}\frac{h}{2}(y_3 + 4\,y_4 + y_5).$$

The sum of the two is

$$A = h\left(\frac{y_1 + 4\,y_2 + 2\,y_3 + 4\,y_4 + y_5}{12}\right).$$

By using a sufficiently large number of ordinates in Simpson's formula, the result can be made as accurate as desired.

Example. Find ln 5 by Simpson's rule. Since

$$\ln 5 = \int_1^5 \frac{dx}{x},$$

we take $y = \dfrac{1}{x}$ in Simpson's formula. Dividing the interval into 4 parts we get

$$\ln 5 = 4\left(\frac{1 + 4\cdot\frac{1}{2} + 2\cdot\frac{1}{3} + 4\cdot\frac{1}{4} + \frac{1}{5}}{12}\right) = 1.622.$$

If we divide the interval into 8 parts, we get

$$\ln 5 = \tfrac{4}{24}\left(1 + \tfrac{8}{3} + \tfrac{2}{2} + \tfrac{8}{5} + \tfrac{2}{3} + \tfrac{8}{7} + \tfrac{2}{4} + \tfrac{8}{9} + \tfrac{1}{5}\right) = 1.6108.$$

The value correct to 4 decimals is

$$\ln 5 = 1.6094.$$

151. Integration in Series. — In calculating integrals it is sometimes convenient to expand a function in infinite series and then integrate the series. This is particularly the case when the integral contains constants for which numerical values are not assigned. For the process to be valid all series used should converge.

Example. Find the length of a quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let a be greater than b. Introduce a parameter ϕ by the equation

$$x = a \sin \phi.$$

Substituting this value in the equation of the ellipse, we find

$$y = b \cos \phi.$$

Using these values of x and y we get

$$s = \int \sqrt{dx^2 + dy^2} = \int_0^{\frac{\pi}{2}} \sqrt{a^2 - (a^2 - b^2) \sin^2 \phi}\, d\phi.$$

This is an elliptic integral. It cannot be represented by an expression containing only a finite number of elementary functions. We therefore express it as an infinite series. By the binomial theorem

$$\sqrt{a^2 - (a^2 - b^2) \sin^2 \phi}$$
$$= a\left[1 - \frac{1}{2}\,\frac{a^2 - b^2}{a^2} \sin^2 \phi - \frac{1}{2.4}\left(\frac{a^2 - b^2}{a^2}\right)^2 \sin^4 \phi \;.\;.\;.\right].$$

Since

$$\int_0^{\frac{\pi}{2}} \sin^2 \phi\, d\phi = \frac{\pi}{4}, \qquad \int_0^{\frac{\pi}{2}} \sin^4 \phi\, d\phi = \frac{3\,\pi}{16},$$

we find by integrating term by term

$$s = a\left[\frac{\pi}{2} - \frac{\pi}{8}\,\frac{a^2 - b^2}{a^2} - \frac{3\,\pi}{128}\left(\frac{a^2 - b^2}{a^2}\right)^2 \;.\;.\;.\right]$$
$$= \frac{\pi a}{2}\left[1 - \frac{a^2 - b^2}{4\,a^2} - \frac{3}{64}\left(\frac{a^2 - b^2}{a^2}\right)^2 \;.\;.\;.\right].$$

If a and b are nearly equal, the value of s can be calculated very rapidly from the series.

EXERCISES

1. Compute

$$\ln 10 = \int_1^{10} \frac{dx}{x}$$

approximately, using the trapezoidal rule with $\Delta x = 1$, and compare with the value obtained by integration.

2. Compute

$$\int_0^1 \sqrt{1 - x^2}\, dx$$

approximately, using the trapezoidal rule with $\Delta x = 0.2$, and compare with the value obtained by integration.

3. Determine the error when the value of the integral

$$\int_1^5 x^4\,dx$$

is found by the prismoidal formula.

4. Show that the prismoidal formula gives the correct volume in each of the following cases: (a) sphere, (b) cone, (c) cylinder, (d) pyramid, (e) segment of a sphere, (f) truncated cone or pyramid.

In each of the following cases compare the value given by the prismoidal formula with that obtained by integration:

5. Area bounded by $y = \sqrt{x}$, $y = 0$, $x = 1$, $x = 5$.

6. Arc of the curve $y = \frac{1}{2}(e^x + e^{-x})$, from $x = 0$ to $x = 1$.

7. Volume generated by rotating one arch of the curve $y = \sin x$ about the x-axis.

Compute each of the following definite integrals using Simpson's rule with four intervals:

8. $\displaystyle\int_0^1 \frac{dx}{1 + x^2}$.

9. $\displaystyle\int_2^4 (x^2 - 4)^{\frac{3}{2}}\,dx$.

10. $\displaystyle\int_0^{\frac{\pi}{6}} \sqrt{\sin x}\,dx$.

11. $\displaystyle\int_2^6 \frac{dx}{\ln x}$.

12. Find the area of the surface generated by rotating the ellipse

$$x^2 + 4\,y^2 = 4$$

about the x-axis.

By expanding into series find the values of

13. $\displaystyle\int_0^1 \sin(x^2)\,dx$.

14. $\displaystyle\int_0^{\frac{\pi}{6}} \sqrt{\cos x}\,dx$.

15. $\displaystyle\int^2 e^x\,dx$.

16. Express

$$\int_0^1 \frac{\sin(\lambda x)}{x}\,dx$$

as a series in powers of λ.

17. Find the length of a quadrant of the ellipse

$$3\,x^2 + 4\,y^2 = 12.$$

•

CHAPTER XX

DIFFERENTIAL EQUATIONS

152. Definitions. — A differential equation is an equation containing differentials or derivatives. Thus

$$(x^2 + y^2)\,dx + 2\,xy\,dy = 0,$$
$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2$$

are differential equations.

A *solution* of a differential equation is an equation connecting the variables such that the derivatives or differentials calculated from that equation satisfy the differential equation. Thus $y = x^2 - 2\,x$ is a solution of the second equation above; for when $x^2 - 2\,x$ is substituted for y the equation is satisfied.

A differential equation containing only a single independent variable, and so containing only total derivatives, is called an *ordinary* differential equation. An equation containing partial derivatives is called a *partial* differential equation. We shall consider only ordinary differential equations in this book.

The *order* of a differential equation is the order of the highest derivative occurring in it.

153. Illustrations of Differential Equations. — Whenever an equation connecting derivatives or differentials is known, the equation connecting the variables can be determined by solving the differential equation. A number of simple cases were treated in Chapter I.

The fundamental problem of integral calculus is to find the function

$$y = \int f(x)\,dx,$$

when $f(x)$ is given. This is equivalent to solving the differential equation

$$dy = f(x)\,dx.$$

Often the slope of a curve is known as a function of x and y,

$$\frac{dy}{dx} = f(x, y).$$

The equation of the curve can be found by solving the differential equation.

In mechanical problems the velocity or acceleration of a particle may be known in terms of the distance s the particle has moved and the time t,

$$\frac{ds}{dt} = v, \qquad \frac{d^2s}{dt^2} = a.$$

The position s can be determined as a function of the time by solving the differential equation.

In physical or chemical problems the rates of change of the variables may be known as functions of the variables and the time. The values of those variables at any time can be found by solving the differential equations.

Example. Find the curve in which the cable of a suspension bridge hangs

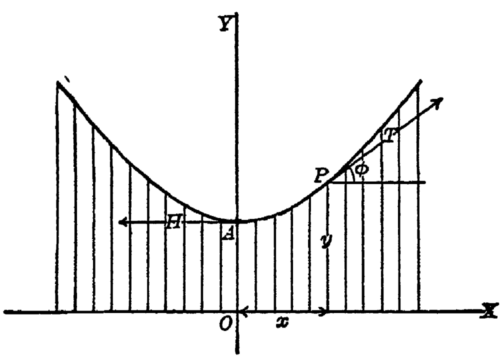

FIG. 153.

Let the bridge be the x-axis and let the y-axis pass through the center of the cable. The portion of the cable AP is in

equilibrium under three forces, a horizontal tension H at A, a tension PT in the direction of the cable at P, and the weight of the portion of the bridge between A and P. The weight of the cable, being very small in comparison with that of the bridge, is neglected.

The weight of the part of the bridge between A and P is proportional to x. Let it be Kx. Since the vertical components of force must be in equilibrium

$$T \sin \phi = Kx.$$

Similarly, from the equilibrium of horizontal components, we have

$$T \cos \phi = H.$$

Dividing the former equation by this, we get

$$\tan \phi = \frac{K}{H} x.$$

But $\tan \phi = \dfrac{dy}{dx}$. Hence

$$\frac{dy}{dx} = \frac{K}{H} x.$$

The solution of this equation is

$$y = \frac{K}{2H} x^2 + c.$$

The curve is therefore a parabola.

154. Constants of Integration. Particular and General Solutions. — To solve the equation

$$\frac{dy}{dx} = f(x),$$

we integrate once and so obtain an equation with one arbitrary constant,

$$y = \int f(x)\, dx + c.$$

To solve the equation

$$\frac{d^2y}{dx^2} = f(x)$$

we integrate twice. The result

$$y = \int \int f(x)\, dx^2 + c_1x + c_2$$

contains two arbitrary constants. Similarly, the integral of the equation

$$\frac{d^ny}{dx^n} = f(x)$$

contains n arbitrary constants.

These illustrations belong to a special type. The rule indicated is, however, general. *The complete, or general, solution of a differential equation of the nth order in two variables contains n arbitrary constants.* If particular values are assigned to any or all of these constants, the result is still a solution. Such a solution is called a *particular* solution.

In most problems leading to differential equations the result desired is a particular solution. To find this we usually find the general solution and then determine the constants from some extra information contained in the statement of the problem.

Example 1. Show that

$$x^2 + y^2 - 2\,cx = 0$$

is the general solution of the differential equation

$$y^2 - x^2 - 2\,xy\frac{dy}{dx} = 0.$$

Differentiating $x^2 + y^2 - 2\,cx = 0$, we get

$$2\,x + 2\,y\frac{dy}{dx} - 2\,c = 0,$$

whence

$$\frac{dy}{dx} = \frac{c - x}{y}.$$

Substituting this value in the differential equation, it becomes

$$y^2 - x^2 - 2\,xy\frac{dy}{dx} = y^2 - x^2 - 2\,x\,(c - x) = y^2 + x^2 - 2\,cx = 0.$$

Hence $x^2 + y^2 - 2\,cx = 0$ is a solution. Since it contains one constant and the differential equation is one of the first order, it is the general solution.

Ex. 2. Find the differential equation of which $y = c_1e^x + c_2e^{2x}$ is the general solution.

Since the given equation contains two constants, the differential equation is one of the second order. We therefore differentiate twice and so obtain

$$\frac{dy}{dx} = c_1e^x + 2\,c_2e^{2x},$$

$$\frac{d^2y}{dx^2} = c_1e^x + 4\,c_2e^{2x}.$$

Eliminating c_1, we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2\,c_2e^{2x},$$

$$\frac{dy}{dx} - y = c_2e^{2x}.$$

Hence

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2\left(\frac{dy}{dx} - y\right)$$

or

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2\,y = 0.$$

This is an equation of the second order having $y = c_1e^x + c_2e^{2x}$ as solution. It is the differential equation required.

EXERCISES

In each of the following exercises, show that the equation given is a solution of the differential equation and state whether it is the general or a particular solution.

1. $y = ce^x + e^{-x}$, $\qquad \dfrac{d^2y}{dx^2} = y.$

2. $x^2 - y^2 = cx,$ $\qquad (x^2 + y^2)\,dx - 2\,xy\,dy = 0.$

3. $y = ce^x \sin x,$ $\qquad \dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2\,y = 0.$

4. $y = c_1 + c_2 \sin (x + c_3),$ $\qquad \dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = 0.$

Find the differential equation of which each of the following equations is the general solution:

5. $y = c_1x + \dfrac{c_2}{x}.$

6. $y = cxe^x.$

7. $y = c_1 \sin x + c_2 \cos x.$

8. $x^2y = c_1 + c_2 \ln x + c_3x^3.$

9. $x^2 + c_1xy + c_2y^2 = 0.$

155. Differential Equations of the First Order in Two Variables. — By solving for $\dfrac{dy}{dx}$ an equation of the first order in two variables x and y can be reduced to the form

$$\frac{dy}{dx} = f(x, y).$$

To solve this equation is equivalent to finding the curves with slope equal to $f(x, y)$. The solution contains one

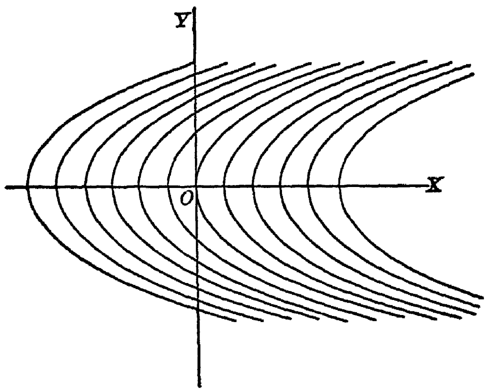

FIG. 155.

arbitrary constant. There is consequently an infinite number of such curves, usually one through each point of the plane.

We cannot always solve even this simple type of equation. In the following articles some cases will be discussed

which frequently occur and for which general methods of solution are known.

156. Variables Separable. — A differential equation of the form

$$M\,dx + N\,dy = 0$$

is called separable if each of the coefficients M and N contains only one of the variables or is the product of a function of x and a function of y. By division the x's and dx can be brought together in the first term, the y's and dy in the second. The two terms can then be integrated separately and the sum of the integrals equated to a constant.

Example 1. $(1 + x^2)\,dy - xy\,dx = 0$.

Dividing by $(1 + x^2)\,y$, this becomes

$$\frac{dy}{y} = \frac{x\,dx}{1 + x^2},$$

whence

$$\ln y = \tfrac{1}{2} \ln (1 + x^2) + c.$$

If $c = \ln k$, this is equivalent to

$$\ln y = \ln \sqrt{1 + x^2} + \ln k = \ln k \sqrt{1 + x^2},$$

and so

$$y = k \sqrt{1 + x^2},$$

where k is an arbitrary constant.

Ex. 2. Find the curve in which the area bounded by the curve, coördinate axes, and a variable ordinate is proportional to the arc forming part of the boundary

Let A be the area and s the length of arc. Then

$$A = ks.$$

Differentiating with respect to x,

$$\frac{dA}{dx} = k\frac{ds}{dx},$$

or

$$y = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{\sqrt{y^2 - k^2}}{k},$$

whence

$$\frac{dy}{\sqrt{y^2 - k^2}} = \frac{dx}{k}.$$

The solution of this is

$$\ln\left(y + \sqrt{y^2 - k^2}\right) = \frac{x}{k} + c.$$

Therefore

$$y + \sqrt{y^2 - k^2} = e^{\frac{x}{k}+c} = e^c e^{\frac{x}{k}} = c_1 e^{\frac{x}{k}},$$

where c_1 is a new constant. Transposing y and squaring, we get

$$y^2 - k^2 = \left(c_1 e^{\frac{x}{k}}\right)^2 - 2\, c_1 e^{\frac{x}{k}} y + y^2.$$

Hence, finally,

$$y = \frac{c_1}{2} e^{\frac{x}{k}} + \frac{k^2}{2\, c_1} e^{-\frac{x}{k}}.$$

157. Exact Differential Equations. — An equation

$$du = 0,$$

obtained by equating to zero the total differential of a function u of x and y, is called an *exact* differential equation. The solution of such an equation is

$$u = c.$$

The condition that $M\,dx + N\,dy$ be an exact differential is (Art. 122)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{157}$$

This equation, therefore, expresses the condition that

$$M\,dx + N\,dy = 0$$

be an exact differential equation.

An exact equation can often be solved by inspection. To find u it is merely necessary to obtain a function whose total differential is $M\,dx + N\,dy$.

If this cannot be found by inspection, it can be determined from the fact that

$$du = M\,dx + N\,dy$$

and so

$$\frac{\partial u}{\partial x} = M.$$

By integrating with y constant, we therefore get

$$u = \int M\,dx + f(y).$$

Since y is constant in the integration, the constant of integration may be a function of y. This function can be found by equating the total differential of u to $M\,dx + N\,dy$. Since $df(y)$ gives terms containing y only, $f(y)$ *can usually be found by integrating the terms in* $N\,dy$ *that do not contain* x. In exceptional cases this may not give the correct result. The answer should, therefore, be tested by differentiation.

Example 1. $(2x - y)\,dx + (4y - x)\,dy = 0.$

The equation is equivalent to

$$2x\,dx + 4y\,dy - (y\,dx + x\,dy) = d(x^2 + 2y^2 - xy) = 0.$$

It is therefore exact and its solution is

$$x^2 + 2y^2 - xy = c.$$

Ex. 2. $(\ln y - 2x)\,dx + \left(\frac{x}{y} - 2y\right)dy = 0.$

In this case

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(\ln y - 2x) = \frac{1}{y},$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{y} - 2y\right) = \frac{1}{y}.$$

These derivatives being equal, the equation is exact. Its solution is

$$x \ln y - x^2 - y^2 = c.$$

The part $x \ln y - x^2$ is obtained by integrating $(\ln y - 2x)\,dx$ with y constant. The term $-y^2$ is the integral of $-2y\,dy$, which is the only term in $\left(\frac{x}{y} - 2y\right)dy$ that does not contain x.

158. Integrating Factors. — If an equation of the form $M\,dx + N\,dy = 0$ is not exact it can be made exact by multiplying by a proper factor. Such a multiplier is called an *integrating factor*.

For example, the equation

$$x\,dy - y\,dx = 0$$

is not exact. But if it is multiplied by $\frac{1}{x^2}$, it takes the form

$$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right) = 0$$

which is exact. It also becomes exact when multiplied by $\frac{1}{y^2}$ or $\frac{1}{xy}$. The functions $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}$ are all integrating factors of $x\,dy - y\,dx = 0$.

While an equation of the form $M\,dx + N\,dy = 0$ always has integrating factors, there is no general method of finding them.

Example 1. $y(1 + xy)\,dx - x\,dy = 0$.

This equation can be written

$$y\,dx - x\,dy + xy^2\,dx = 0.$$

Dividing by y^2,

$$\frac{y\,dx - x\,dy}{y^2} + x\,dx = 0.$$

Both terms of this equation are exact differentials. The solution is

$$\frac{x}{y} + \frac{1}{2}x^2 = c.$$

Ex. 2. $(y^2 + 2\,xy)\,dx + (2\,x^2 + 3\,xy)\,dy = 0.$

This is equivalent to

$$y^2\,dx + 3\,xy\,dy + 2\,xy\,dx + 2\,x^2\,dy = 0.$$

Multiplying by y, it becomes

$$y^3\,dx + 3\,xy^2\,dy + 2\,xy^2\,dx + 2\,x^2y\,dy = d\,(xy^3 + x^2y^2) = 0.$$

Hence

$$xy^3 + x^2y^2 = c.$$

159. Linear Equations. — A differential equation of the form

$$\frac{dy}{dx} + Py = Q, \tag{159a}$$

where P and Q are functions of x or constants, is called *linear*. The linear equation is one of the first degree in one of the variables (y in this case) and its derivative. Any functions of the other variable can occur.

If the linear equation is written in the form (159a),

$$e^{\int P\,dx}$$

is an integrating factor; for when multiplied by this factor the equation becomes

$$e^{\int P\,dx}\frac{dy}{dx} + ye^{\int P\,dx}P = e^{\int P\,dx}Q.$$

The left side is the derivative of

$$ye^{\int P\,dx}$$

Hence

$$ye^{\int P\,dx} = \int e^{\int P\,dx}Q\,dx + c \tag{159b}$$

is the solution.

Example 1. $\dfrac{dy}{dx} + \dfrac{2}{x}y = x^3.$

In this case

$$\int P\,dx = \int \frac{2}{x}\,dx = 2\ln x = \ln x^2.$$

Hence

$$e^{\int P\,dx} = e^{\ln x^2} = x^2.$$

The integrating factor is, therefore, x^2. Multiplying by x^2 and changing to differentials, the equation becomes

$$x^2\,dy + 2\,xy\,dx = x^5\,dx.$$

The integral is

$$x^2y = \tfrac{1}{6}\,x^6 + c.$$

Ex. 2. $(1 + y^2)\,dx - (xy + y + y^3)\,dy = 0.$

This is an equation of the first degree in x and dx. Dividing by $(1 + y^2)\,dy$, it becomes

$$\frac{dx}{dy} - \frac{y}{1 + y^2}x = y.$$

P is here a function of y and

$$\int P\,dy = \int \frac{-y\,dy}{1 + y^2} = -\frac{1}{2}\ln(1 + y^2) = \ln\frac{1}{\sqrt{1 + y^2}},$$

$$e^{\int P\,dy} = \frac{1}{\sqrt{1 + y^2}}.$$

Multiplying by the integrating factor, the equation becomes

$$\frac{dx}{\sqrt{1 + y^2}} - \frac{xy\,dy}{(1 + y^2)^{\frac{3}{2}}} = \frac{y\,dy}{\sqrt{1 + y^2}},$$

whence

$$\frac{x}{\sqrt{1 + y^2}} = \sqrt{1 + y^2} + c$$

and

$$x = 1 + y^2 + c\sqrt{1 + y^2}.$$

160. Equations Reducible to Linear Form. — An equation of the form

$$\frac{dy}{dx} + Py = Qy^n, \tag{160}$$

where P and Q are functions of x, can be made linear by a change of variable. Dividing by y^n, it becomes

$$y^{-n}\frac{dy}{dx}+Py^{-n+1}=Q.$$

If we take

$$y^{1-n}=u$$

as a new variable, the equation takes the form

$$\frac{1}{1-n}\frac{du}{dx}+Pu=Q,$$

which is linear.

Example. $\dfrac{dy}{dx}+\dfrac{2}{x}y=\dfrac{y^3}{x^3}.$

Division by y^3 gives

$$y^{-3}\frac{dy}{dx}+\frac{2}{x}y^{-2}=\frac{1}{x^3}.$$

Let

$$u=y^{-2}.$$

Then

$$\frac{du}{dx}=-2y^{-3}\frac{dy}{dx},$$

whence

$$y^{-3}\frac{dy}{dx}=-\frac{1}{2}\frac{du}{dx}.$$

Substituting these values, we get

$$-\frac{1}{2}\frac{du}{dx}+\frac{2}{x}u=\frac{1}{x^3},$$

and so

$$\frac{du}{dx}-\frac{4}{x}u=-\frac{2}{x^3}.$$

This is a linear equation with solution

$$u=\frac{1}{3x^2}+cx^4,$$

or, since $u=y^{-2}$,

$$\frac{1}{y^2}=\frac{1}{3x^2}+cx^4.$$

161. Homogeneous Equations. — A function $f(x, y)$ is said to be a homogeneous function of the nth degree if

$$f(tx, ty) = t^n f(x, y).$$

Thus $\sqrt{x^2 + y^2}$ is a homogeneous function of the first degree; for

$$\sqrt{x^2t^2 + y^2t^2} = t\sqrt{x^2 + y^2}.$$

It is easily seen that a polynomial whose terms are all of the nth degree is a homogeneous function of the nth degree.

The differential equation

$$M\,dx + N\,dy = 0$$

is called homogeneous if M and N are homogeneous functions of the same degree. To solve a homogeneous equation substitute

$$y = vx.$$

The new equation will be separable.

Example 1. $x\dfrac{dy}{dx} - y = \sqrt{x^2 + y^2}.$

This is a homogeneous equation of the first degree. Substituting $y = vx$, it becomes

$$x\left(v + x\frac{dv}{dx}\right) - vx = \sqrt{x^2 + v^2x^2}$$

whence

$$x\,\frac{dv}{dx} = \sqrt{1 + v^2}.$$

This is a separable equation with solution

$$x = c\left(v + \sqrt{1 + v^2}\right).$$

Replacing v by $\dfrac{y}{x}$, transposing, squaring, etc., the equation becomes

$$x^2 - 2\,cy = c^2.$$

Ex. 2. $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0.$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2+y^2}}{y},$$

or

$$y\,dy + x\,dx = \pm\sqrt{x^2+y^2}\,dx.$$

This is a homogeneous equation of the first degree. It is much easier, however, to divide by $\sqrt{x^2+y^2}$ and integrate at once. The result is

$$\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = \pm\,dx,$$

whence

$$\sqrt{x^2+y^2} = c \pm x$$

and

$$y^2 = c^2 \pm 2\,cx.$$

Since c may be either positive or negative, the answer can be written

$$y^2 = c^2 + 2\,cx.$$

162. Change of Variable. — We have solved the homogeneous equation by taking as new variable

$$v = \frac{y}{x}.$$

It may be possible to reduce any equation to a simpler form by taking some function u of x and y as a new variable or by taking two functions u and v as new variables. Such functions are often suggested by the equation. In other cases they may be indicated by the problem in the solution of which the equation occurs.

Example. $(x-y)^2\frac{dy}{dx} = a^2.$

Let $x - y = u$. Then

$$1 - \frac{dy}{dx} = \frac{du}{dx}$$

and the differential equation becomes

$$u^2\left(1 - \frac{du}{dx}\right) = a^2,$$

whence

$$u^2 - a^2 = u^2\frac{du}{dx}.$$

The variables are separable. The solution is

$$x = u + \frac{a}{2}\ln\frac{u-a}{u+a} + c$$

$$= x - y + \frac{a}{2}\ln\frac{x-y-a}{x-y+a} + c,$$

or

$$y = \frac{a}{2}\ln\frac{x-y-a}{x-y+a} + c.$$

EXERCISES

Solve the following differential equations:

1. $x^3\,dy - y^3\,dx = 0.$
2. $\tan x \sin^2 y\,dx + \cos^2 x \cot y\,dy = 0.$
3. $(xy^2 + x)\,dx + (y - x^2y)\,dy = 0.$
4. $(xy^2 + x)\,dx + (x^2y - y)\,dy = 0.$
5. $(3x^2 + 2xy - y^2)\,dx + (x^2 - 2xy - 3y^2)\,dy = 0.$
6. $x\frac{dy}{dx} - y = y^3.$
7. $x\,dx + y\,dy = a(x^2 + y^2)\,dy.$
8. $x\frac{dy}{dx} + y = y^2.$
9. $\frac{dy}{dx} - ay = e^{bx}.$
10. $x^2\frac{dy}{dx} - 2xy = 3.$
11. $x^2\frac{dy}{dx} - 2xy = 3y.$
12. $(2xy^2 - y)\,dx + x\,dy = 0.$
13. $(1 - x^2)\frac{dy}{dx} + 2xy = (1 - x^2)^2.$
14. $\tan x\frac{dy}{dx} - y = a.$
15. $x\frac{dy}{dx} - 3y + x^4y^2 = 0.$
16. $\frac{dy}{dx} + y = xy^3.$

17. $(x^2 - 1)^{\frac{3}{2}}\, dy + (x^3 + 3\, xy \sqrt{x^2 - 1})\, dx = 0.$
18. $x\, dx + (x + y)\, dy = 0.$
19. $(x^2 + y^2)\, dx - 2\, xy\, dy = 0.$
20. $y\, dx + (x + y)\, dy = 0$
21. $(x^3 - 3\, x^2 y)\, dx + (y^3 - x^3)\, dy = 0.$
22. $ye^y\, dx = (y^3 + 2\, xe^y)\, dy.$
23. $\left(xye^{\frac{x}{y}} + y^2\right) dx - x^2 e^{\frac{x}{y}}\, dy = 0.$
24. $(x + y - 1)\, dx + (2\, x + 2\, y - 3)\, dy = 0.$
25. $3\, y^2 \dfrac{dy}{dx} - y^3 = x.$
26. $e^y \left(\dfrac{dy}{dx} + 1\right) = e^x.$
27. $x \left(\dfrac{dy}{dx}\right)^2 - 2\, y \dfrac{dy}{dx} - x = 0.$
28. $\left(\dfrac{dy}{dx}\right)^2 - (x + y) \dfrac{dy}{dx} + xy = 0.$
29. $y^2 \left(\dfrac{dy}{dx}\right)^2 + 2\, xy \dfrac{dy}{dx} - y^2 = 0.$

30. The differential equation for the charge q of a condenser having a capacity C connected in series with a circuit of resistance R is

$$R \frac{dq}{dt} + \frac{q}{C} = E,$$

where E is the electromotive force. Find v as a function of t if E is constant and $q = 0$ when $t = 0$.

31. The differential equation for the current induced by an electromotive force $E \sin \alpha t$ in a circuit having the inductance L and resistance R is

$$L \frac{di}{dt} + Ri = E \sin \alpha t.$$

Solve for i and determine the constants so that $i = I$ when $t = 0$.

Let PT be the tangent and PN the normal to a plane curve at $P\ (x, y)$ (Fig. 162a). Determine the curve or curves in each of the following cases:

32. The subtangent $TM = 3$ and the curve passes through (2, 2).
33. The subnormal $MN = a$ and the curve passes through (0, 0).
34. The intercept OT of the tangent on the x-axis is one-half the abscissa OM.
35. The length PT of the tangent is a constant a.
36. The length PN of the normal is a constant a.
37. The perpendicular from M to PT is a constant a.

Using polar coördinates (Fig. 162b), find the curve or curves in each of the following cases:

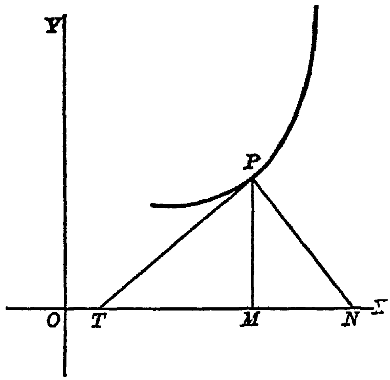

FIG. 162a.

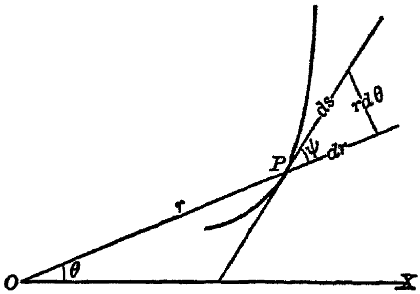

FIG. 162b.

38. The curve passes through (1, 0) and makes with OP a constant angle $\psi = \frac{\pi}{4}$.

39. The angles ψ and θ are equal.

40. The distance from O to the tangent is a constant a.

41. The projection of OP on the tangent at P is a constant a.

42. Find the curve passing through (0, 1) in which the area bounded by the curve, x-axis, a fixed, and a variable ordinate is proportional to that ordinate.

43. Find the curve in which the length of arc is proportional to the angle between the tangents at its end.

44. Find the curve in which the length of arc is proportional to the difference of the abscissas at its ends.

45. Find the curve in which the length of any arc is proportional to the angle it subtends at a fixed point.

46. Find the curve in which the length of arc is proportional to the difference of the distances of its ends from a fixed point.

47. Oxygen flows through one tube into a liter flask filled with air while the mixture of oxygen and air escapes through another. If the action is so slow that the mixture in the flask may be considered uniform, what percentage of oxygen will the flask contain after 10 liters of gas have passed through? (Assume that air contains 21 per cent by volume of oxygen.)

163. Certain Equations of the Second Order. — There are two forms of the second order differential equation that

occur in mechanical problems so frequently that they deserve special attention. These are

$$(1)\qquad \frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right),$$

$$(2)\qquad \frac{d^2y}{dx^2} = f\left(y, \frac{dy}{dx}\right).$$

The peculiarity of these equations is that one of the variables (y in the first, x in the second) does not appear directly in the equation. They are both reduced to equations of the first order by the substitution

$$\frac{dy}{dx} = p.$$

This substitution reduces the first equation to the form

$$\frac{dp}{dx} = f\,(x, p).$$

This is a first order equation whose solution has the form

$$p = F\,(x, c_1),$$

or, since $p = \dfrac{dy}{dx}$,

$$\frac{dy}{dx} = F\,(x, c_1).$$

This is again an equation of the first order. Its solution is the result required.

In case of an equation of the second type, write the second derivative in the form

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}.$$

The differential equation then becomes

$$p\frac{dp}{dy} = f\,(y, p).$$

Solve this for p and proceed as before.

Example 1. $(1+x^2)\dfrac{d^2y}{dx^2}+1+\left(\dfrac{dy}{dx}\right)^2=0.$

Substituting p for $\dfrac{dy}{dx}$, we get

$$(1+x^2)\frac{dp}{dx}+1+p^2=0.$$

This is a separable equation with solution

$$p=\frac{c_1-x}{1+c_1x},$$

whence

$$dy=\frac{c_1-x}{1+c_1x}\,dx.$$

The integral of this is

$$y=-\frac{x}{c_1}+\frac{c_1^2+1}{c_1^2}\ln(1+c_1x)+c_2.$$

By a change of constants this becomes

$$y=cx+(1+c^2)\ln(c-x)+c'.$$

Ex. 2. $y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=1.$

Substituting

$$\frac{dy}{dx}=p,\qquad \frac{d^2y}{dx^2}=p\frac{dp}{dy},$$

we get

$$yp\frac{dp}{dy}+p^2=1.$$

The solution of this is

$$y^2p^2=y^2+c_1.$$

Replacing p by $\dfrac{dy}{dx}$ and solving again, we get

$$y^2+c_1=(x+c_2)^2.$$

Ex. 3. Under the action of gravitation the acceleration of a falling body is $\frac{k}{r^2}$, where k is constant and r the distance from the center of the earth. Find the time required for the body to fall to the earth from a distance equal to that of the moon.

Let r_1 be the radius of the earth (about 4000 miles), r_2 the distance from the center of the earth to the moon (about 240,000 miles) and g the acceleration of gravity at the surface of the earth (about 32 feet per second). At the surface of the earth $r = r_1$ and

$$a = \frac{k}{r_1^2} = -g.$$

The negative sign is used because the acceleration is toward the origin ($r = 0$). Hence $k = -gr_1^2$ and the general value of the acceleration is

$$a = \frac{v\,dv}{dr} = -\frac{gr_1^2}{r^2},$$

where v is the velocity. The solution of this equation is

$$v^2 = \frac{2\,gr_1^2}{r} + C.$$

When $r = r_2$, $v = 0$. Consequently,

$$C = -2\,g\frac{r_1^2}{r_2^2}$$

and

$$v = \frac{dr}{dt} = -\sqrt{2\,gr_1^2\left(\frac{1}{r} - \frac{1}{r_2}\right)}.$$

The time of falling is therefore

$$t = \int_{r_1}^{r_2}\sqrt{\frac{rr_2}{2\,gr_1^2\,(r_2 - r)}}\,dr = 116 \text{ hours.}$$

This result is obtained by using the numerical values of r_1 and r_2 and reducing g to miles per hour.

164. Linear Differential Equations with Constant Coefficients. — A differential equation of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = f(x), \quad (164a)$$

where $a_1, a_2, \cdots, a_n$ are constants, is called a linear differential equation with constant coefficients. For practical applications this is the most important type of differential equation.

In discussing these equations we shall find it convenient to represent the operation $\frac{d}{dx}$ by D. Then

$$\frac{dy}{dx} = Dy, \qquad \frac{d^2y}{dx^2} = D^2y, \text{ etc.}$$

Equation (164a) can be written

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)y = f(x). \quad (164b)$$

This signifies that if the operation

$$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n \quad (164c)$$

is performed on y the result will be $f(x)$. The operation consists in differentiating y, n times, $n - 1$ times, $n - 2$ times, etc., multiplying the results by 1, a_1, a_2, etc., and adding.

With the differential equation is associated an algebraic equation

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n = 0.$$

If the roots of this *auxiliary* equation are $r_1, r_2, \cdots, r_n$, the polynomial (164c) can be factored in the form

$$(D - r_1)(D - r_2) \cdots (D - r_n). \quad (164d)$$

If we operate on y with $D - a$, we get

$$(D - a)\, y = \frac{dy}{dx} - ay.$$

If we operate on this with $D - b$, we get

$$(D - b) \cdot (D - a)\, y = (D - b)\left(\frac{dy}{dx} - ay\right)$$
$$= \frac{d^2y}{dx^2} - (a + b)\frac{dy}{dx} + ab.$$

The same result is obtained by operating on y with

$$(D - a)\,(D - b) = D^2 - (a + b)\, D + ab.$$

Similarly, if we operate in succession with the factors of (164d) we get the same result that we should get by operating directly with the product (164c).

165. Equation with Right Hand Member Zero. — To solve the equation

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)\, y = 0 \qquad (165a)$$

factor the symbolic operator and so reduce the equation to the form

$$(D - r_1)\,(D - r_2) \cdots (D - r_n)\, y = 0.$$

The value $y = c_1e^{r_1x}$ is a solution; for

$$(D - r_1)\, c_1e^{r_1x} = c_1r_1e^{r_1x} - r_1c_1e^{r_1x} = 0$$

and the equation can be written

$$(D-r_2) \cdots (D-r_n) \cdot (D-r_1)\, y = (D-r_2) \cdots (D-r_n) \cdot 0 = 0.$$

Similarly, $y = c_2e^{r_2x}$, $y = c_3e^{r_3x}$, etc., are solutions. Finally

$$y = c_1e^{r_1x} + c_2e^{r_2x} + \cdots + c_ne^{r_nx} \qquad (165b)$$

is a solution; for the result of operating on y is the sum of the results of operating on $c_1e^{r_1x}$, $c_2e^{r_2x}$, etc., each of which is zero.

If the roots $r_1, r_2, \cdots, r_n$ are all different, (165b) contains n constants and so is the complete solution of (165a). If, however, two roots r_1 and r_2 are equal.

$$c_1e^{r_1x} + c_2e^{r_1x} = (c_1 + c_2)\, e^{r_1x}$$

contains only one arbitrary constant $c_1 + c_2$ and (165b) contains only $n - 1$ arbitrary constants. In this case, however, $xe^{r_1 x}$ is also a solution; for

$$(D - r_1)\, xe^{r_1 x} = r_1 xe^{r_1 x} + e^{r_1 x} - r_1 xe^{r_1 x} = e^{r_1 x}$$

and so

$$(D - r_1)^2\, xe^{r_1 x} = (D - r_1)\, e^{r_1 x} = 0.$$

If then two roots r_1 and r_2 are equal, the part of the solution corresponding to these roots is

$$(c_1 + c_2 x)e^{r_1 x}.$$

More generally, if m roots $r_1, r_2, \ldots r_m$ are equal, the part of the solution corresponding to them is

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_m x^{m-1})e^{r_1 x}. \tag{165c}$$

If the coefficients $a_1, a_2, \ldots a_n$, are real, imaginary roots occur in pairs

$$r_1 = \alpha + \beta\sqrt{-1}, \qquad r_2 = \alpha - \beta\sqrt{-1}.$$

The terms $c_1 e^{r_1 x}$, $c_2 e^{r_2 x}$ are imaginary but they can be replaced by two other terms that are real. Using these values of r_1 and r_2, we have

$$(D - r_1)\,(D - r_2) = (D - \alpha)^2 + \beta^2.$$

By performing the differentiations it can easily be verified that

$$[(D - \alpha)^2 + \beta^2] \cdot e^{\alpha x} \sin \beta x = 0,$$
$$[(D - \alpha)^2 + \beta^2] \cdot e^{\alpha x} \cos \beta x = 0.$$

Therefore

$$e^{\alpha x}\,[c_1 \sin \beta x + c_2 \cos \beta x] \tag{165d}$$

is a solution. This function, in which α and β are real, can, therefore, be used as the part of the solution corresponding to two imaginary roots $r = \alpha \pm \beta\sqrt{-1}$.

To solve the differential equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)\, y = 0,$$

let $r_1, r_2, \ldots, r_n$ *be the roots of the auxiliary equation*

$$r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n = 0.$$

If these roots are all real and different, the solution of the equation is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}.$$

If m of the roots $r_1, r_2, \ldots, r_m$ are equal, the corresponding part of the solution is

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_m x^{m-1})\, e^{r_1 x}.$$

The part of the solution corresponding to two imaginary roots $r = \alpha \pm \beta\sqrt{-1}$ is

$$e^{\alpha x}\,[c_1 \sin \beta x + c_2 \cos \beta x].$$

Example 1. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2\,y = 0.$

This is equivalent to

$$(D^2 - D - 2)\, y = 0.$$

The roots of the auxiliary equation

$$r^2 - r - 2 = 0$$

are -1 and 2. Hence the solution is

$$y = c_1 e^{-x} + c_2 e^{2x}.$$

Ex. 2. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 3\,y = 0.$

The roots of the auxiliary equation

$$r^3 + r^2 - 5\,r + 3 = 0$$

are 1, 1, -3. The part of the solution corresponding to the two roots equal to 1 is

$$(c_1 + c_2 x)\, e^x.$$

Hence

$$y = (c_1 + c_2 x)\, e^x + c_3 e^{-3x}.$$

Example 3. $(D^2 + 2D + 2)y = 0.$

The roots of the auxiliary equation are

$$-1 \pm \sqrt{-1}.$$

Therefore $\alpha = -1, \beta = 1$ in (165d) and

$$y = e^{-x}[c_1 \sin x + c_2 \cos x].$$

166. Equation with Right Hand Member a Function of x. —Let $y = u$ be the *general solution* of the equation

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)y = 0$$

and let $y = v$ be *any solution* of the equation

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots a_n)y = f(x). \qquad (166)$$

Then $$y = u + v$$

is a solution of (166); for the operation

$$D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n$$

when performed on u gives zero and when performed on v gives $f(x)$. Furthermore, $u + v$ contains n arbitrary constants. Hence it is the general solution of (166).

The part u is called the *complementary function*, v the *particular integral.* To solve an equation of the form (166), first solve the equation with right hand member zero and then add to the result any solution of (166).

A particular integral can often be found by inspection. If not, the general form of the integral can usually be determined by the following rules:

1. If $f(x) = ax^n + a_1x^{n-1} + \cdots + a_n$, assume
$$y = Ax^n + A_1x^{n-1} + \cdots + A_n.$$

But, if 0 occurs m times as a root in the auxiliary equation, assume

$$y = x^m[Ax^n + A_1x^{n-1} + \cdots + A_m].$$

2. If $f(x) = ce^{ax}$, assume

$$y = Ae^{ax}.$$

But, if a occurs m times as a root of the auxiliary equation, assume

$$y = Ax^m e^{ax}.$$

3. If $f(x) = a \cos \beta x + b \sin \beta x$, assume

$$y = A \cos \beta x + B \sin \beta x.$$

But, if $\cos \beta x$ and $\sin \beta x$ occur in the complementary function, assume

$$y = x[A \cos \beta x + B \sin \beta x].$$

4. If $f(x) = ae^{\alpha x} \cos \beta x + be^{\alpha x} \sin \beta x$, assume

$$y = Ae^{\alpha x} \cos \beta x + \beta e^{\alpha x} \sin \beta x.$$

But, if $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ occur in the complementary function, assume

$$y = xe^{\alpha x}[A \cos \beta x + B \sin \beta x].$$

If $f(x)$ contains terms of different types, take for y the sum of the corresponding expressions. Substitute the value of y in the differential equation and determine the constants so that the equation is satisfied.

Example 1. $\dfrac{d^2y}{dx^2} + 4y = 2x + 3.$

A particular solution is evidently

$$y = \tfrac{1}{4}(2x + 3).$$

Hence the complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \tfrac{1}{4}(2x + 3).$$

Ex. 2. $(D^2 + 3D + 2)y = 2 + e^x.$

Substituting $y = A + Be^x$, we get

$$2A + 6Be^x = 2 + e^x.$$

Hence

$$2A = 2, \qquad 6B = 1$$

and

$$y = 1 + \tfrac{1}{6}e^x + c_1e^{-x} + c_2e^{-2x}.$$

Ex. 3. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = x^2.$

The roots of the auxiliary equation are 0, 0, -1. Since 0 is twice a root, we assume

$$y = x^2(Ax^2 + Bx + C) = Ax^4 + Bx^3 + Cx^2.$$

Substituting this value,

$$12Ax^2 + (24A + 6B)x + 6B + 2C = x^2.$$

Consequently,

$$12A = 1, \qquad 24A + 6B = 0, \qquad 6B + 2C = 0,$$

whence

$$A = \tfrac{1}{12}, \qquad B = -\tfrac{1}{3}, \qquad C = 1.$$

The solution is

$$y = \tfrac{1}{12}x^4 - \tfrac{1}{3}x^3 + x^2 + c_1 + c_2x + c_3e^{-x}.$$

167. Simultaneous Equations. — We consider only linear equations with constant coefficients containing one independent variable and as many dependent variables as equations. All but one of the dependent variables can be eliminated by a process analogous to that used in solving linear algebraic equations. The one remaining dependent variable is the solution of a linear equation. Its value can be found and the other functions can then be determined by substituting this value in the previous equations.

Example.
$$\frac{dx}{dt} + 2x - 3y = t,$$
$$\frac{dy}{dt} - 3x + 2y = e^{2t}.$$

Using D for $\dfrac{d}{dt}$, these equations can be written

$$(D + 2)x - 3y = t,$$
$$(D + 2)y - 3x = e^{2t}.$$

To eliminate y, multiply the first equation by $D + 2$ and the second by 3. The result is

$$(D + 2)^2x - 3(D + 2)y = 1 + 2t,$$
$$3(D + 2)y - 9x = 3e^{2t}.$$

Adding, we get

$$[(D+2)^2-9]\,x = 1+2\,t+3\,e^{2\,t}.$$

The solution of this equation is

$$x = -\tfrac{2}{5}\,t - \tfrac{13}{25} + \tfrac{3}{7}\,e^{2\,t} + c_1e^t + c_2e^{-5\,t}.$$

Substituting this value in the first equation, we find

$$y = \tfrac{1}{3}\,(D+2)\,x - \tfrac{1}{3}\,t = -\tfrac{3}{5}\,t - \tfrac{12}{25} + \tfrac{4}{7}\,e^{2\,t} + c_1e^t - c_2e^{-5\,t}.$$

EXERCISES

Solve the following equations:

1. $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+x=0.$

2. $(x+1)\dfrac{d^2y}{dx^2}-(x+2)\dfrac{dy}{dx}+x+2=0.$

3. $\dfrac{d^2y}{dx^2}=a^2y.$

4. $\dfrac{d^2y}{dx^2}=-a^2y.$

5. $\dfrac{d^2s}{dt^2}=-\dfrac{k}{s^2}.$

6. $\dfrac{d^2s}{dt^2}+a^2\left(\dfrac{ds}{dt}\right)^2=b^2.$

7. $x\dfrac{d^2y}{dx^2}=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}.$

8. $y\dfrac{d^2y}{dx^2}=1+\left(\dfrac{dy}{dx}\right)^2.$

9. $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}=0.$

10. $\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}-6\,y=0.$

11. $\dfrac{d^2y}{dx^2}-6\dfrac{dy}{dx}+9\,y=0.$

12. $\dfrac{d^2y}{dx^2}+y=0.$

13. $\dfrac{d^3y}{dx^3}-2\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}=0.$

14. $\dfrac{d^4y}{dx^4}=y.$

15. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+3\,y=0.$

16. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=0.$

17. $\dfrac{d^4y}{dx^4}-3\dfrac{d^2y}{dx^2}+2\,y=0.$

18. $\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}-y=0.$

19. $\dfrac{d^2y}{dx^2}+y=x+3.$

20. $\dfrac{d^2y}{dx^2}-4\,y=e^x.$

21. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}-6\,y=x^2.$

22. $\dfrac{dy}{dx}-y=\sin x.$

23. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}=2\,x-3.$

24. $\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+5\,y=x+e^{3x}.$

25. $\dfrac{d^2y}{dx^2}-a^2y=e^{ax}.$

26. $\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}+y=\cos 2\,x.$

27. $\dfrac{d^3y}{dx^3}-y=x^3-x^2.$

28. $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+3\,y=e^{2x}\sin x.$

29. $\frac{d^2y}{dx^2} - 9y = e^{3x}\cos x.$

30. $\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} = \cos 4x.$

31. $\frac{d^2y}{dx^2} + 4y = \cos 2x.$

32. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^x + e^{-x}.$

33. $\frac{dy}{dt} + x = e^t, \quad \frac{dx}{dt} - y = e^{-t}.$

34. $\frac{dx}{dt} = x - 2y + 1, \quad \frac{dy}{dt} = x - y + 2.$

35. $4\frac{dx}{dt} - \frac{dy}{dt} + 3x = \sin t, \quad \frac{dx}{dt} + y = \cos t.$

36. $\frac{d^2y}{dt^2} = x, \quad \frac{d^2x}{dt^2} = y.$

37. Solve the equation

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$$

and determine the constants so that $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$.

38. Solve $\frac{d^2y}{dx^2} = 3\sqrt{y}$ under the hypothesis that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$.

39. When a body sinks slowly in a liquid, its acceleration and velocity approximately satisfy the equation

$$a = g - kv,$$

g and k being constants. Find the distance passed over as a function of the time if the body starts from rest.

40. The acceleration and velocity of a body falling in the air approximately satisfy the equation $a = g - kv^2$, g and k being constants. Find the distance traversed as a function of the time if the body falls from rest.

41. A weight supported by a spiral spring is lifted a distance b and let fall. Its acceleration is given by the equation $a = -k^2s$, k being constant and s the displacement from the position of equilibrium. Find s in terms of the time t.

42. Find the velocity with which a meteor strikes the earth, assuming that it starts from rest at an indefinitely great distance and moves toward the earth with an acceleration inversely proportional to the square of its distance from the center.

43. A body falling in a hole through the center of the earth would have an acceleration toward the center proportional to its distance from the center. If the body starts from rest at the surface, find the time required to fall through.

44. A chain 5 feet long starts with one foot of its length hanging over the edge of a smooth table. The acceleration of the chain will be proportional to the amount over the edge. Find the time required to slide off.

45. A chain hangs over a smooth peg, 8 feet of its length being on one side and 10 on the other. Its acceleration will be proportional to the difference in length of the two sides. Find the time required to slide off.

ANSWERS TO EXERCISES

Page 7

1. $y = x \pm \sqrt{x-1}$.
4. $f(y, x) = y^2 - 2\,xy$.
5. $\frac{3}{5}$.
6. 7.
7. $\dfrac{1}{2\sqrt{a}}$.
8. 1.

Page 13

1. .0434.
2. .5000.
7. $(\frac{3}{2}, -\frac{3}{4})$.
8. $(0, 0), (\pm 1, -1)$.

Page 15

1. 48 ft./sec., 96 ft./sec.
2. 164.
3. $3\,t^2 + 4\,t$.
4. $t = 2$.
5. $2\,\pi an$.

Pages 23, 24

1. $6\,x - 2$.
2. $4\,x(x^2 - 3\,x + 3)$.
3. $x^2 + 2\,x$.
4. $\dfrac{3\,(x+1)}{\sqrt{x}}$.
5. $a^2 + 4\,ax - 3\,x^2$.
6. $\dfrac{6\,x - 1}{2\sqrt{x}}$.
7. $(3\,x - 1)\,(9\,x - 1)$.
8. $x(3x-2)(2x+3)^2(36x+41)$.
9. $\dfrac{2\,x}{(1-x^2)^2}$.
10. $\dfrac{8}{(2\,x+3)^2}$.
11. $-\dfrac{6\,x}{(x-1)^4}$.
12. $\dfrac{1-x}{\sqrt{1+2\,x-x^2}}$.
13. $-\dfrac{a^2}{x^2\sqrt{a^2-x^2}}$.
14. $\dfrac{a^2 - 2\,x^2}{\sqrt{a^2-x^2}}$.
15. $\dfrac{x - \sqrt{a^2+x^2}}{a\sqrt{a^2+x^2}}$.
16. $\dfrac{2\,x}{(x^2+1)\sqrt{x^4-1}}$.
17. $\dfrac{4\,ac - b^2}{2\,(ax^2+bx+c)^{\frac{3}{2}}}$.
18. $\dfrac{32\,x^4 - 32\,x^2 + 5}{\sqrt{2\,x^2-1}}$.
19. $\dfrac{1}{x^4\sqrt{x^2+1}}$.
20. $\dfrac{9}{125}$.
21. 18.
22. $x = 1, -2, -\frac{2}{3}$.
23. 45°.

Pages 26, 27

1. $1+\frac{1}{x^2},\ -\frac{2}{x^3}$.

2. $\frac{2}{(x+1)^2},\ -\frac{4}{(x+1)^3}$.

3. $x(5x+4)(x+2)^2,\ 4(x+2)(5x^2+8x+2)$.

4. $\frac{a}{2y},\ -\frac{a^2}{4y^3}$.

5. $-\frac{x}{y},\ -\frac{a^2}{y^3}$.

6. $\frac{x}{y},\ -\frac{1}{y^3}$.

7. $-\frac{1}{(x-1)^2},\ \frac{2}{(x-1)^3}$.

8. $\frac{1}{2}\left(1+\frac{1}{x^2}\right),\ -\frac{1}{x^3}$.

9. $\frac{2x-y}{x-2y}$, 0.

12. $t^2\frac{d^3x}{dt^3}$.

Pages 32–34

1. $v = 16$, $a = -32$. Rises until $t = 2\frac{1}{2}$. Highest point $h = 120$.
2. 8.
3. The velocity decreases until $t = 2$. The speed increases during that interval.
6. $\omega = 200\,\pi$ rad./min. The speed of the belt is $400\,\pi$ ft./min.
7. The angular velocity of the smaller pulley is twice that of the larger.
9. $\frac{10}{r}$.
10. $2\,\pi rv$.
11. $-\frac{2}{3}\,\pi$.
12. 4 mi./hr.
13. $\frac{1}{3}\sqrt{3}$.
14. 2.31 in./sec.
16. $8\frac{1}{3}$ ft./sec., $3\frac{1}{3}$ ft./sec.
18. 17.89 mi./hr.
19. 7.56 mi./hr.
20. 110.8 ft./sec.
21. 11.55 ft.

Pages 41–44

1. Minimum value 2.
2. Maximum at $x = -2$, minimum at $x = 1$.
3. Minima at $x = \pm 1$, maximum at $x = 0$.
4. Minimum value 0.
12. $\frac{2}{3}\,a\sqrt{3}$.
13. Altitude $\frac{4}{3}\,a$.
14. The side of the base is twice the depth of the box.
16. The altitude is $\frac{4}{\pi}$ times the diameter of the base.
17. Depth 1 inch, side of base 4 inches.
18. 3.22 cu. ft.
19. The arc is equal to twice the radius.
20. Length 24 inches.

21. $\frac{1}{4}(a_1 + a_2 + a_3 + a_4)$.
22. 16.97 in.
24. 8 in.
25. 2.45 in.
26. Radius of semicircle equals height of rectangle.
27. Radius of base equals two-thirds of the altitude.
28. The ratio of length to width is $\frac{a}{b}$.
30. The width at top is twice the slant height of the bank.
31. 418.8 sq. ft.
32. 3690 cu. ft.
33. Altitude $4a$.
34. Its distance from the more intense source is $\sqrt[3]{2}$ times its distance from the other.
35. At the end of four hours.
36. He should walk 4.43 miles
37. 15.53.
38. 20 ft.
41. Speed through the water 7.5 miles per hour.
42. 13.6 knots.

Page 47

1. Max., $x = 0$; min., $x = \frac{2}{3}$.
2. $(-1, 0)$.
3. $(-1, 0)$.
4. The side of the square is zero.

Pages 52–54

1. $12 \cos 3x$.
2. $-\sin \frac{1}{2} x$.
3. $2 \sin \frac{1}{2} x \cos \frac{1}{2} x$.
4. $-\cos^2 x \sin x$.
5. $\sin^2 x$.
6. $5 \cos^3 5x$.
7. 0.
8. $\sin^3 \frac{x}{3}$.
9. $2 \sec^4 (2x)$.
10. $\tan^5 x \sec x$.
11. $-\cot^6 x$.
12. $\frac{3 \sec^2 x (\tan x - 1)}{2\sqrt{\tan x}}$.
13. $x \sin^3 x$.
14. $-\sin^6 x$.
15. $\frac{\cos \frac{1}{2} x}{(1 - \sin \frac{1}{2} x)^2}$.
16. $2 \sec x (\sec x + \tan x)^2$.
18. $A = -3$
19. $A = \frac{2}{3}$, $B = -\frac{1}{3}$.
20. 45°.
21. 5.
22. velocity $= -ak$, acceleration $= 0$.
24. $\frac{80}{3}\pi$ mi./min.
25. 0.2182.
26. 5.034 sq. ft./sec.
28. 90°.
29. 46.87.
30. The needle will be inclined to the horizontal at an angle of approximately 32° 30′.
31. 120°.
32. 120°.
33. $\frac{a}{\pi}$.
34. 7.79 in.
35. 28.28 ft.

Pages 56, 57

1. $\dfrac{3}{\sqrt{6x - 9x^2}}$.
2. $\dfrac{1}{\sqrt{2ax - x^2}}$.
3. $\dfrac{6}{9x^2 + 4}$.
4. $-\dfrac{2}{x^2 + 1}$.
5. $\dfrac{1}{(4x + 1)\sqrt{x}}$.
6. $\dfrac{1}{\sqrt{2 + 2x - 4x^2}}$.
7. $\dfrac{a}{x^2 + a^2}$.
8. -1.
9. $\sqrt{a^2 - x^2}$.
10. $\sqrt{\dfrac{a - x}{a + x}}$.
11. $\dfrac{\sqrt{x^2 - a^2}}{x}$.
12. $\dfrac{16}{(x^2 + 4)^2}$.
13. $\cos^{-1}(2x)$
14. $\sin^{-1} x$.
15. $\dfrac{\sqrt{x^2 - 4}}{x}$
16. $\dfrac{2a^2}{x^3\sqrt{x^2 - a^2}}$
17. $\dfrac{4}{3\cos x + 5}$
18. $\dfrac{1}{5 + 3\cos x}$.
19. $\dfrac{1}{ax^2 + bx + c}$.
20. 0.
23. 12 ft.
24. $\dfrac{v\sqrt{x^2 - a^2}}{ax}$.

Pages 61, 62

1. $\dfrac{6x + 5}{3x^2 + 5x + 1}$.
2. $\dfrac{2}{x - 2}$.
3. $\sec x \csc x$.
4. $\ln x$.
5. $\dfrac{2}{x^2 - 1}$.
6. $\dfrac{0.8686\,(x - 1)}{x^2 - 2x}$.
7. $\sec x$.
8. $\dfrac{1}{\sqrt{x^2 - a^2}}$.
9. $\dfrac{\cos^3 x}{\sin x}$.
10. $\dfrac{1}{2\sqrt{x^2 + 5x + 6}}$.
11. $\csc 3x$.
12. $\frac{1}{2}(e^x - e^{-x})$.
13. $2x + 2^x \ln 2$.
14. $ex^{e-1} + e^x$.
15. $x^{n-1}n^x(n + x\ln n)$.
16. $e^{\sin x}\cos x$.
17. $-9xe^{-3x}$.
18. xe^{2x}.
19. $\dfrac{4}{(e^x + e^{-x})^2}$.
20. x^2e^x.
21. $e^x \sin 2x$.
22. $\dfrac{1}{e^x + 4}$.

23. $\sec^{-1} x$.
24. $\tan^{-1} \frac{1}{2} x$.
25. $\dfrac{8}{x(x^2 - 4)^2}$.
26. $\tan^5 x$.
27. $\dfrac{\cos^2 ax}{\sin ax}$.
28. $\dfrac{1}{x \ln ax}$.
29. $\sin (\ln x)$.
30. $(x^2 + 1)^{\frac{3}{2}}$.
31. 1.1752.
32. $x = \frac{3}{4}\pi + n\pi$, where n is any integer.
33. 0.3679.

Pages 71, 72

5. $(.641)\Delta x$, $(.0604)\Delta x$.
6. $(.0016)\Delta\theta$, $(.000025)\Delta\theta$.
7. $(.035)\dfrac{\pi}{180}$.
8. $(.00036)\Delta x$.
9. $dy = -.01234$, $\Delta y = -.01245$, $\Delta y - dy = -(.006)\Delta\theta$.

Page 77

1. $\tan^2 \theta \, d\theta$.
2. $\dfrac{\sqrt{x^2 - a^2}}{x} dx$.
3. $(2x + y)\,dx + (x - 2y)\,dy$.
4. $\dfrac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}}$.
5. $\dfrac{y\,dx + x\,dy}{xy}$.
6. $\dfrac{x\,dy - y\,dx}{x^2 + y^2}$.
7. $(x + y + 1)e^{x-y}\,dx - (x + y - 1)e^{x-y}\,dy$.
8. $e^{x+y}\,dz + ze^{x+y}(dx + dy)$.
9. $\dfrac{dy}{dx} = -\dfrac{b}{a}\cot t$, $\dfrac{d^2y}{dx^2} = -\dfrac{b}{a^2}\csc^3 t$.
10. $\dfrac{dy}{dx} = \dfrac{e^t + e^{-t}}{e^t - e^{-t}}$, $\dfrac{d^2y}{dt^2} = -\dfrac{8}{(e^t - e^{-t})^3}$.
11. $\dfrac{dy}{dx} = -\dfrac{b}{a}$, $\dfrac{d^2y}{dx^2} = 0$.
12. $\dfrac{dy}{dx} = \csc\theta$, $\dfrac{d^2y}{dx^2} = -\cot^3\theta$.
13. $\dfrac{dy}{dx} = -(\csc\theta - \cot\theta)^2$, $\dfrac{d^2y}{dx^2} = 2(\csc\theta - \cot\theta)^3$.
14. $\dfrac{dy}{dx} = \dfrac{\cos t + \sin t}{\cos t - \sin t}$, $\dfrac{d^2y}{dx^2} = \dfrac{2e^{-t}}{(\cos t - \sin t)^3}$.
15. $\dfrac{d^2y}{dx^2} = \dfrac{2}{(1 + t)^3}$, $\dfrac{d^2x}{dy^2} = \dfrac{2}{(1 - t)^3}$.
16. $\dfrac{dy}{dx} = \dfrac{\sin\theta \dfrac{dr}{d\theta} + r\cos\theta}{\cos\theta \dfrac{dr}{d\theta} - r\sin\theta}$.
17. $\dfrac{d^2y}{dx^2} = -\dfrac{1}{\left(\dfrac{dx}{dy}\right)^3}\dfrac{d^2x}{dy^2}$.

Pages 80, 81

1. Tangent line $2y - x + 5 = 0$, normal $y + 2x = 0$.
2. Tangent line $x + y - 4 = 0$, normal $x - y = 0$.
3. Tangent line $x + 2y + a = 0$, normal $2x - y - 3a = 0$.
4. Tangent line $y = a(1 + x \ln b)$, normal $x = a \ln b(a - y)$.
5. Tangent line $x + y = \frac{\pi}{2}$, normal $x - y = \frac{\pi}{2}$.
6. Tangent line $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$,
   normal $b^2x_1(y - y_1) - a^2y_1(x - x_1) = 0$.
7. Tangent line $y = x - 2$, normal $x + y = 2$.
8. Tangent line $3y + 2x = 1$, normal $2y - 3x = 5$.
9. $y = \frac{1 - \cos \phi_1}{\sin \phi_1}(a\phi_1 - x)$.
10. $\tan^{-1} \frac{3}{4}$, $90°$.
11. $45°$.
12. $45°$.
13. $\tan^{-1}(2\sqrt{2})$.
14. $0°$.
15. $\tan^{-1} \frac{1}{3}$, $\tan^{-1} 3$.

Pages 84, 85

1. Point of inflection $(0, 2)$. Concave upward on the right of this point, downward on the left.

2. Point of inflection $(1, -4)$. Concave upward on the right of this point, downward on the left.

3. Concave upward for all values of x. No point of inflection.

4. Point of inflection $(1, 0)$. Concave upward on the left of this point, downward on the right.

5. Point of inflection $\left(2, \frac{2}{e^2}\right)$. Concave upward on the right of this point, downward on the left.

6. Points of inflection $(\pm\frac{1}{2}\sqrt{2}, e^{-\frac{1}{2}})$. Concave downward between these points, upward outside.

7. Concave upward when $-3 < x < 0$ or $x > 3$, concave downward when $x < -3$ or $0 < x < 3$. Points of inflection $(0, 0)$, $(-3, -1)$, $(3, 1)$.

8. Concave downward for all values of x. No point of inflection.

Pages 91, 92

1. $\sqrt{15}$.
4. $2\sqrt{2}$.
5. $3\sqrt{2}$.
6. 1.
7. $\frac{2}{3} a\sqrt{2}$.
8. $\rho = \frac{1}{4} \frac{(y^2 + 1)^2}{y}$.
9. $\rho = \sec x$.
10. $\rho = \frac{a}{4}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})^2$.
11. $\rho = 4a \sin \frac{1}{2} \phi$.

Pages 94, 95

1. $45°$.
2. $45°$.
3. $60°$.
4. $135°$.
5. $\dfrac{\pi}{6}$.
7. $60°$.
8. $(0, 0)$, $(\frac{3}{2}\,a, \pm\frac{2}{3}\,\pi)$.
10. $\frac{5}{3}\sqrt{3}$.

Pages 102–104

1. The components of velocity are $\dfrac{dx}{dt} = 3$, $\dfrac{dy}{dt} = 4$, and the speed is $\dfrac{ds}{dt} = 5$.

2. $\dfrac{dx}{dt} = e^t(\cos t - \sin t)$,

   $\dfrac{dy}{dt} = e^t(\sin t + \cos t)$,

   $\dfrac{ds}{dt} = \sqrt{2}\,e^t$.

4. $\dfrac{dx}{dt} = \pm\frac{1}{2}\,v$, $\dfrac{dy}{dt} = \mp\frac{1}{2}\,v\sqrt{3}$, $\dfrac{ds}{dt} = v$.

5. $\dfrac{dx}{dt} = \dfrac{dy}{dt} = \pm\frac{1}{2}\,v\sqrt{2}$.

7. The acceleration is toward the center and of magnitude $\dfrac{v^2}{a}$, where v is the speed.

10. The acceleration is parallel to the x-axis and of magnitude $\dfrac{v^2}{2\,a}$.

11. Its velocity is zero and its acceleration of magnitude $a\omega^2$ parallel to the y-axis.

13. The component of velocity is $-2\,a\omega\sin\theta$, and the component of acceleration $-4\,a\omega^2\cos\theta$.

Page 110

1. $\frac{1}{4}\,x^4 - \frac{4}{3}\,x^3 + x^2 + x + C$.
2. $\frac{1}{3}\,x^3 + \dfrac{1}{x} + C$.
3. $\frac{2}{3}\,x^{\frac{3}{2}} + 2\,x^{\frac{1}{2}} + C$.
4. $\dfrac{\sqrt{2\,x}}{3}(2\,x - 3) + C$.
5. $\frac{1}{4}\,x^4 - \frac{2}{3}\,x^3 + \frac{1}{2}\,x^2 + C$.
6. $\frac{1}{6}(2\,x - 1)^3 + C$.
7. $2\,y + 3\ln y + C$.
8. $y - \dfrac{1}{y} + 2\ln y + C$.
9. $\ln(x + 1) + C$.
10. $-\dfrac{1}{x + 1} + C$.
11. $\sqrt{2\,x + 1} + C$.
12. $\frac{1}{2}\ln(x^2 + 2) + C$.

13. $\sqrt{x^2 - 1} + C.$

14. $-\dfrac{1}{4\,(a^2 + x^2)^2} + C.$

15. $-\frac{1}{3}\,(a^2 - x^2)^{\frac{3}{2}} + C.$

16. $\frac{1}{3}\ln\,(a^3 + x^3) + C.$

17. $\frac{2}{9}\,(x^3 - 1)^{\frac{3}{2}} + C.$

18. $\ln\,(x^2 + x + 1) + C.$

19. $2\sqrt{x^2 + ax + b} + C.$

20. $-\dfrac{1}{5\,a}\ln\,(1 - at^5) + C.$

21. $-\frac{1}{5}\,(a^2 - t^2)^{\frac{5}{2}} + C.$

22. $x + 3\ln\,(x - 2) + C.$

23. $\frac{1}{2}\ln(2x^2+1) - \dfrac{1}{4(2x^2+1)} + C.$

24. $\frac{1}{4}\left(1 - \dfrac{1}{x}\right)^4 + C.$

25. $\dfrac{(x^n + a)^{1-n}}{n(1 - n)} + C$, if n is not equal to 0 or 1.

26. $-\frac{2}{3}\left(\dfrac{x+1}{x}\right)^{\frac{3}{2}} + C.$

Pages 116, 117

1. $30\,t + \frac{1}{2}\,gt^2.$
2. $h = 60 + 100\,t - \frac{1}{2}\,gt^2$, maximum height approximately 215 ft.
3. 138.
4. $\frac{1}{2}\,\alpha t^2.$
5. $\dfrac{\omega_0{}^2}{2\,k}.$
6. $y = x - \frac{1}{2}\,x^2 - \frac{1}{2}.$
7. $y = \frac{1}{6}\,x^3 - \frac{1}{2}\,x^2 + \frac{1}{2}\,x + \frac{13}{6}.$
8. $x^2 = y.$
9. $y^2 = 4\,x.$
10. $N = N_0e^{kt}$, where N_0 is the number at the start.
12. 17.1 min.

Page 122

1. $\frac{1}{3}\sin 3\,x - \frac{1}{2}\cos 2\,x + C.$
2. $\frac{5}{2}\cos\left(\dfrac{3 - 2\,x}{5}\right) + C.$
3. $3\tan\frac{1}{3}\,\theta + C.$
4. $-2\csc\frac{1}{2}\,\theta + C.$
5. $\frac{1}{2}\sin^2\theta + C.$
6. $\tan x + C.$
7. $-\frac{1}{2}\cot 2\,x + C.$
8. $-\csc x + C.$
9. $\frac{1}{3}\sec^3 x + C.$
10. $\ln\,(1 + \sin x) + C.$
11. $\csc x - \cot x + C.$
12. $-\frac{1}{2}\cos\,(x^2 + 1) + C.$
13. $\frac{1}{2}\,[\tan 2\,x + \sec 2\,x] + C.$
14. $x + \tan x + 2\ln\,(\sec x + \tan x) + C.$
15. $-2\sin x + \ln\,(\csc x - \cot x) + C.$
16. $\frac{1}{4}\sin^4 x + C.$
17. $\frac{1}{3}\tan^3 x + C.$
18. $\frac{1}{3}\sec^3 x + C.$
19. $-\frac{1}{5}\cos^5 x + C.$
20. $-\frac{1}{2}\ln\,(1 + 2\cot x) + C.$
21. $\sin^{-1}\frac{1}{2}\,x + C.$
22. $\frac{1}{2}\sin^{-1}\dfrac{2\,x}{\sqrt{3}} + C.$
23. $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2\,x}{\sqrt{3}} + C.$
24. $\dfrac{1}{\sqrt{3}}\sec^{-1}\dfrac{x\sqrt{2}}{\sqrt{3}} + C.$
25. $\dfrac{1}{2\sqrt{3}}\tan^{-1}\dfrac{2\,y}{\sqrt{3}} + C.$
26. $\frac{1}{3}\ln\,(3\,t + \sqrt{9\,t^2 + 1}) + C.$
27. $\frac{2}{3}\sec^{-1}\dfrac{2\,x}{3} + C.$
28. $3\sin^{-1}\dfrac{x}{2} - 2\sqrt{4 - x^2} + C.$

29. $\sqrt{x^2+4} + \ln(x + \sqrt{x^2+4}) + C.$
30. $\ln(\sin x + \sqrt{1+\sin^2 x}) + C.$
31. $\sqrt{1+\sin^2 x} + C.$
32. $\tan^{-1}(\sin x) + C.$
33. $2\sqrt{1-\cos\theta} + C.$
34. $\ln(1+\ln x) + C.$
35. $\frac{1}{2}\sin^{-1}\frac{x^2}{a^2} + C.$
36. $-\frac{1}{2}e^{-x^2} + C.$
37. $\frac{1}{2a}[e^{2ax} - e^{-2ax}] - 2x + C.$
38. $\frac{1}{2}\ln(1+e^{2x}) + C.$
39. $\tan^{-1}(e^x) + C.$
40. $-\sin^{-1}(e^{-t}) + C.$

Page 124

1. $\frac{1}{2}\tan^{-1}\frac{x+3}{2} + C.$
2. $\frac{1}{2}\sin^{-1}\frac{2x-1}{2} + C.$
3. $\frac{1}{\sqrt{3}}\ln\left[(x+1)\sqrt{3} + \sqrt{3x^2+6x+2}\right] + C.$
4. $\frac{1}{\sqrt{3}}\sin^{-1}\frac{(x-1)\sqrt{3}}{\sqrt{5}} + C.$
5. $\frac{1}{2}\sec^{-1}\frac{x-1}{2} + C.$
6. $\sec^{-1}(2x-1) + C.$
7. $\frac{1}{8}\ln(4x^2-4x+2) - \frac{1}{4}\tan^{-1}(2x-1) + C.$
8. $\frac{2}{3}\sqrt{3x^2-6x-1} + \frac{1}{\sqrt{3}}\ln\left[(x-1)\sqrt{3} + \sqrt{3x^2-6x-1}\right] + C.$
9. $\frac{1}{2}\ln(x^2+2x+2) - \tan^{-1}(x+1) + C.$
10. $\frac{1}{4}\ln\left(2x+1+\sqrt{4x^2+4x-1}\right) + \frac{1}{4\sqrt{2}}\sec^{-1}\frac{2x+1}{\sqrt{2}} + C.$
11. $-\frac{3}{\sqrt{x^2-2x+3}} + C.$
12. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{e^x+1}{\sqrt{2}}\right) + C.$

Pages 129, 130

1. $\frac{1}{3}\cos^3 x - \cos x + C.$
2. $\frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x + \sin x + C.$
3. $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$
4. $\frac{1}{12}\sin^4 3\theta - \frac{1}{18}\sin^6 3\theta + C.$
5. $\cos\theta - \frac{2}{3}\cos^3\theta + C.$
6. $-\cos x - \frac{1}{2}\cos^2 x + C.$
7. $\cos x + \ln(\csc x - \cot x) + C.$
8. $\ln\sin x - \frac{1}{2}\sin^2 x + C.$
9. $\tan x + \frac{1}{3}\tan^3 x + C.$

10. $-(\cot y + \frac{2}{3}\cot^3 y + \frac{1}{5}\cot^5 y) + C.$
11. $\tan x - x + C.$
12. $\frac{2}{3}\sec^3 \frac{1}{2}x + C.$
13. $\frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C.$
14. $-\frac{1}{2}\cot^2 x - \ln\sin x + C.$
15. $-\frac{1}{3}\cot^3 x - \frac{1}{5}\cot^5 x + C.$
16. $\ln(\csc 2x - \cot 2x) + C.$
17. $-\frac{1}{2}\cos x - \frac{1}{6}\cos 3x + C.$
18. $\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + C.$
19. $\frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C.$
20. $\frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x + C.$
21. $\frac{1}{2}x - \frac{1}{8}\sin 4x + C.$
22. $\frac{1}{2}x + \frac{1}{16}\sin 8x + C.$
23. $\frac{1}{16}x - \frac{1}{48}\sin^3 2x - \frac{1}{64}\sin 4x + C.$
24. $\frac{1}{16}x + \frac{1}{24}\sin^3 x - \frac{1}{32}\sin 2x + C.$
25. $\frac{5}{16}x - \frac{1}{4}\sin 2x + \frac{1}{48}\sin^3 2x + \frac{3}{64}\sin 4x + C.$
26. $-\cot\frac{1}{2}x + C.$
27. $\tan x - \sec x + C$
28. $2\sqrt{2}\sin\frac{\theta}{2} + C.$
29. $\frac{1}{3}(x^2 - a^2)^{\frac{3}{2}} + C.$
30. $\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln(x + \sqrt{x^2 - a^2}) + C.$
31. $\frac{1}{15}(3x^2 - 2a^2)(x^2 + a^2)^{\frac{3}{2}} + C.$
32. $\frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln(x + \sqrt{x^2 + a^2}) + C.$
33. $\frac{x}{2}\sqrt{x^2 + a^2} - \frac{a^2}{2}\ln(x + \sqrt{x^2 + a^2}) + C.$
34. $-\dfrac{x}{a^2\sqrt{x^2 - a^2}} + C.$
35. $\frac{1}{a}\ln\dfrac{a - \sqrt{a^2 - x^2}}{x} + C.$
36. $\dfrac{1}{\sqrt{a^2 - x^2}}.$

Page 135

1. $\frac{1}{2}x^2 + 4x - 2\ln(x - 1) + 12\ln(x - 2) + C.$
2. $\ln\dfrac{x^2}{x + 1} + C.$
3. $\ln\dfrac{\sqrt{x^2 - 1}}{x} + C.$
4. $x - \ln x + \frac{3}{2}\ln(x - 2) - \frac{1}{2}\ln(x + 2) + C.$
5. $x + \dfrac{1}{x} + \ln\dfrac{(x - 1)^2}{x} + C.$
6. $\dfrac{1}{x + 1} + \ln(x + 1) + C.$

7. $\dfrac{x}{2(1-x^2)} + \frac{1}{4}\ln\dfrac{x+1}{x-1} + C.$

8. $\ln\dfrac{x-2}{x} - \dfrac{1}{x^2} + C.$

9. $\ln\dfrac{x}{\sqrt{x^2+1}} + \tan^{-1} x + C.$

10. $\ln\dfrac{(x-1)^2}{x^2+2x+1} + C.$

11. $\frac{1}{8}\ln\dfrac{x^2-2}{x^2+2} + C.$

12. $x + \ln\dfrac{x+1}{x^2+2x+2} - 2\tan^{-1}(x+1) + C.$

13. $\dfrac{2}{x^2+4} + \frac{1}{2}\ln(x^2+4) + C.$

14. $-[x + 4\sqrt{x} + 4\ln(1-\sqrt{x})] + C.$

15. $\frac{2}{3}(x-2)\sqrt{x+1} + C.$

16. $\frac{2}{15}(3x+2a)(x-a)^{\frac{3}{2}} + C.$

17. $2\sqrt{x+2} - 2\tan^{-1}\sqrt{x+2} + C.$

18. $\frac{4}{3}x^{\frac{3}{4}} - 4x^{\frac{1}{4}} + 4\tan^{-1}(x^{\frac{1}{4}}) + C.$

Pages 138, 139

1. $x\sin x + \cos x + C.$
2. $(x-1)e^x + C.$
3. $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$
4. $x\sin^{-1} x + \sqrt{1-x^2} + C.$
5. $x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C.$
6. $x\ln(x+\sqrt{a^2+x^2}) - \sqrt{a^2+x^2} + C.$
7. $\frac{1}{2}x^2\sec^{-1} x - \frac{1}{2}\sqrt{x^2-1} + C.$
8. $-(x^3+3x^2+6x+6)e^{-x} + C.$
9. $2(x-1)\sin x + (1+2x-x^2)\cos x + C.$
10. $\dfrac{x}{2}\sqrt{x^2-a^2} - \dfrac{a^2}{2}\ln(x+\sqrt{x^2-a^2}) + C.$
11. $\dfrac{x}{2}\sqrt{a^2+x^2} + \dfrac{a^2}{2}\ln(x+\sqrt{a^2+x^2}) + C.$
12. $\frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C.$
13. $\frac{1}{2}e^x(\sin x + \cos x) + C.$
14. $\frac{1}{5}[3\sin 2x\sin 3x + 2\cos 2x\cos 3x] + C.$
15. $\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln(\sec x + \tan x) + C.$
16. $\frac{1}{8}x(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3}{8}a^4\sin^{-1}\dfrac{x}{a} + C.$

Pages 150, 151

1. 4.05.
2. 2.829.
3. 0.1121.
4. Exact area $\frac{1}{3}$.

10. $\frac{7}{8}$.
11. 0.6931.
12. $\frac{1}{2}$.
13. $\frac{2}{3}$.
14. 1.
15. 4.
16. $\frac{1}{2}$.
17. $2\tan^{-1} 2$.
18. $\frac{8}{3}$.
19. $\dfrac{\pi}{2}$.
20. $\frac{1}{2}\sqrt{2}$.

Pages 155, 156

1. 2.
2. 4.5.
3. $\frac{4}{3}$.
4. 32.
5. 4.5.
6. $20\frac{5}{6}$.
7. $\frac{10}{3}$.
8. 5.4.
9. 1.955.
10. $\frac{64}{3}$.
11. πab.
12. $2\pi + \frac{4}{3}$, $6\pi - \frac{4}{3}$.
13. 4π.
14. 2π.
15. $a^2[2\sqrt{3} - \ln(2 + \sqrt{3})]$.
16. $3\pi a^2$.
17. $\frac{1}{2}a^2\phi$.

Pages 158, 159

1. πa^2.
2. $\frac{1}{4}\pi a^2$.
3. $\frac{2}{3}a^2\sqrt{3}$.
4. $\frac{1}{2}a^2$.
5. $\frac{3}{2}\pi$.
6. $\frac{3}{2}\pi a^2$.
7. $\frac{8}{3}a^2$.
8. $16\pi^3a^2$.
11. $\frac{1}{8}a^2(\pi - 2)$.
12. $\frac{1}{8}a^2(\pi - 2)$.
13. $\frac{1}{4}\pi a^2$.
14. $\frac{1}{8}a^2(9\sqrt{3} - 2\pi)$.
15. $\frac{1}{2}a^2$.

Pages 162, 163

3. $\frac{16}{15}\pi$.
4. $\frac{112}{15}\pi a^3$.
5. $\frac{16}{5}\pi a^3$.
6. $\frac{32}{3}\pi a^3$.
7. $\frac{4}{3}\pi(a^2 - b^2)^{\frac{3}{2}}$.
8. 9π.
9. $\frac{1}{4}\pi^2$.
10. $\frac{4}{3}\pi ab^2$.
11. $5\pi^2a^3$.
12. $\frac{8}{3}\pi a^3$.

Pages 166, 167

2. $\frac{8}{3}a^3$.
3. a^2h.
4. $\frac{4}{3}\pi ab^2$.
5. $\frac{2}{3}a^3\tan\alpha$.
6. $\frac{1}{3}(3\pi - 2)a^2h$.
7. $\frac{1}{3}a^3\sin 2\alpha$.
8. $\frac{1}{2}\pi a^3\sin 2\alpha$.
9. $\dfrac{16a^3}{3\sin\alpha}$.
10. $\frac{4}{3}a^2h$.
11. $\frac{1}{4}k\pi^2a^2$.

Pages 173, 174

1. 9.073.
2. 54.56.
3. $\ln(2 + \sqrt{3})$.
4. 1.096.
5. $6a$.
6. $a\left(e - \dfrac{1}{e}\right)$.
7. $8a$.
8. $2\pi^2a$.
11. $21.26\,k$.
12. $2\pi a$.
13. $\frac{4}{3}a\sqrt{3}$.
15. $2a[\sqrt{2} + \ln(1 + \sqrt{2})]$.
16. $8a$.
17. $\frac{3}{2}\pi a$.

Pages 176, 177

1. $4\,\pi a^2$.
2. $\pi a\sqrt{a^2 + h^2}$.
3. $\dfrac{31\,\pi}{5}$.
4. $\frac{12}{5}\,\pi\sqrt{3}$.
5. $2\,\pi b\left[b + \dfrac{a^2}{\sqrt{a^2 - b^2}}\sin^{-1}\dfrac{\sqrt{a^2 - b^2}}{a}\right]$.
6. $\frac{12}{5}\,\pi a^2$.
7. $\frac{64}{3}\,\pi a^2$.
8. $2\,\pi a\left[a + \dfrac{b^2}{\sqrt{a^2 - b^2}}\ln\dfrac{a + \sqrt{a^2 - b^2}}{b}\right]$
9. $\pi a\left[\sqrt{a^2 + 2\,b^2} + \dfrac{b^2}{\sqrt{a^2 + b^2}}\ln\left(\dfrac{\sqrt{a^2 + b^2} + \sqrt{a^2 + 2\,b^2}}{b}\right)\right]$.
10. $\pi^2 a^2$.
11. $\frac{2}{5}\,\pi\,\sqrt{2}\,(e^{\pi} - 2)$.
12. $8\,\pi a^2(1 - \frac{1}{2}\,\sqrt{2})$.

Pages 179, 180

1 $\frac{4}{3}\,\pi[a^3 - (a^2 - b^2)^{\frac{3}{2}}]$.
2. $\frac{4}{3}\,\pi a^3(2\,\sqrt{2} - 1)$.
3. $\frac{2}{3}\,\pi a^3(1 - \cos\alpha)$.
4. $\frac{4}{3}\,\pi^3 a^2$.
5. $8\,a$.
6. $3\,\pi a^2$.
7. $2\,a^2$.
8. $16\,a^2$.
9. $8\,a^2$.
10. $\pi a^2\sqrt{2}$.
11. $\frac{32}{9}\,a^3$.
12. $\pi a^2\sin\alpha$.

Pages 182, 183

1. 20,000 lbs.
2. 15,000 lbs.
3. $\frac{1}{3}\,wbh^2$.
4. $\frac{1}{6}\,wbh^2$.
5. 2830 lbs.
6. $\frac{2}{3}\,wab^2$.
7. $\frac{4}{5}\,wbh^2$.
8. 13,100 tons.
9. 22,089 lbs.
10. 1600 lbs.
12. $\frac{1}{4}\,\sqrt{3}\,wa^2b$.

Pages 194–196

1. $\frac{1}{2}\,pb^2h$.
2. $\frac{1}{12}\,bwh^3$.
3. 281,000 ton-ft.
4. $(\frac{8}{3}, \frac{11}{6})$.

5. On the line joining the centers, 4 inches from the center of the larger cube.

6. On the radius perpendicular to the diameter of the semicircle at distance $\dfrac{2\,a}{\pi}$ from the center.

7. (3.088, 0).

8. $\frac{1}{3}\,h$.

9. At distance $\dfrac{4\,a}{3\,\pi}$ from each of the bounding radii.

10. $\frac{3}{5}a, 0)$.

11. $(0, \frac{2}{5})$.

12. $(\frac{9}{20}a, \frac{9}{20}a)$.

13. $\left(0, \frac{4b}{3\pi}\right)$.

14. The distances from the outer edges to the center of gravity are $2\frac{3}{7}$ and $9\frac{9}{28}$ inches respectively.

15. On the axis of symmetry at distance $\frac{4(b^3 - a^3)}{3\pi(b^2 - a^2)}$ from the center.

16. On the bisector at distance $\frac{2a \sin \alpha}{3\alpha}$ from the center.

17. $(\pi a, \frac{5}{6}a)$.

18. $(\pi a, \frac{2}{3}a)$.

19. On the axis at distance $\frac{3}{8}a$ from the center.

20. On the axis at distance one-fourth the altitude from the base.

21. On the axis at distance $\frac{5}{8}h$ from the base.

22. $(\frac{4}{3}a, 0)$.

23. $(0, \frac{5}{6}a\sqrt{2})$.

24. On the axis of the cone at distance $\frac{1}{2}a$ from the base.

25. On the axis at distance $\frac{3}{8}\left[\frac{b^4 - a^4}{b^3 - a^3}\right]$ from the center.

26. On the axis at distance $\frac{1}{2}a$ from the center.

27. On the axis at distance $\frac{1}{3}h$ from the base.

28. $\frac{3}{8}a$.

29. $\frac{5}{8}a \tan \alpha$.

Pages 197–199

1. $\frac{1}{3}ab^3$.
2. $\frac{1}{4}bh^3$.
3. $\frac{1}{12}bh^3$.
4. 39.01.
5. $\frac{8}{7}a^4$.
6. $\frac{1}{4}Ma^2$.
7. $\frac{1}{3}Ml^2$.
8. $\frac{1}{2}Ma^2$.
9. $\frac{\pi}{2}a^4$.
11. $\frac{1}{4}\pi ab(a^2 + b^2)$.
12. $\frac{1}{2}Ma^2$.
13. $\frac{1}{2}M(r_1^2 + r_2^2)$.
14. $\frac{2}{5}\pi r^4 h$.
15. $\frac{64}{3}\pi a^5$.
16. $\frac{2}{5}Ma^2$.
17. $\frac{1}{2}\pi^2 a^2 b(3a^2 + 4b^2)$.
18. $\frac{1}{2}Ma^2$.
19. $\frac{2}{3}Ma^2$.
20. $\frac{1}{4}Ma^2\omega^2$.
21. $\frac{1}{5}Ma^2\omega^2$.

Pages 201–203

1. $\frac{1}{2}k\frac{(b-a)^2}{a}$.
2. 282 ft.-lbs.
3. 463 ft.-lbs.
5. 25,100 ft.-lbs.
6. $\frac{1}{2}k(a^2 - b^2)$.
7. 211,000,000 ft.-lbs.
8. 106,000,000 ft.-lbs.
9. $\frac{2}{3}\pi\mu aW$, where a is the radius of the shaft.
10. $\frac{k}{2\pi h}\ln\frac{b}{a}$.
11. $\frac{k(b-a)}{4\pi ab}$.

Pages 208–210

1. $\frac{1}{4}\pi a$.
2. $\dfrac{2}{\pi}$.
3. $\frac{1}{2} gt$.
4. $\frac{2}{3} gt$.
5. 37.6 lbs./sq. in.
6. $\frac{1}{2} RA^2$. Equivalent direct current $\dfrac{A}{\sqrt{2}}$ amperes.
8. $\dfrac{1}{b}$.
10. $2\pi^2 a^2 b$.
11. 1.94 cu. in.
12. $\frac{32}{3}\pi a^2$.
14. $\frac{2}{3}\pi a^3(1 - \cos\alpha)$.
15. $\frac{1}{4}\pi a^3(5\pi + 16)$.
16. Its distance from the center of the sphere is $\frac{3}{8} a(1 + \cos\alpha)$.

Page 216

21. $(-2, 1, 0)$.
22. $(1, 1, 2)$.

Pages 221, 222

1. Increment 23, principal part 22
2. Increment -0.379, principal part $-0\ 385$.
3. Increment 55.64, principal part 54.00.
4. 0.343.
5. 1%.
6. 3%.
7. 0.7%.
8. $(2x + y)\,dx + (x - 2y)\,dy$.
9. $\sin(x + y)(y\,dx + x\,dy) + xy\cos(x + y)(dx + dy)$.
10. $y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$.
11. $\left(\dfrac{1}{y} - \dfrac{z}{x^2}\right)dx + \left(\dfrac{1}{z} - \dfrac{x}{y^2}\right)dy + \left(\dfrac{1}{x} - \dfrac{y}{z^2}\right)dz$.

Pages 229, 230

1. $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}$.
2. $\dfrac{\partial f}{\partial x} + 2x\dfrac{\partial f}{\partial z}$.
3. $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} + 2\dfrac{\partial f}{\partial z}$.
4. $\left(\dfrac{\partial u}{\partial x}\right)_y = \dfrac{\partial f}{\partial x}, \quad \left(\dfrac{\partial u}{\partial x}\right)_r = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}, \quad \left(\dfrac{\partial u}{\partial x}\right)_s = \dfrac{\partial f}{\partial x} + 2\dfrac{\partial f}{\partial y}$.
5. $\dfrac{dz}{dx} = \dfrac{\dfrac{\partial f}{\partial y} - \dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z}}$.

Pages 239, 240

1. $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-1}{2}$.
2. $\dfrac{x-\sqrt{2}}{\sqrt{2}} = \dfrac{y-1}{2} = z - \dfrac{\pi}{4}$.
3. $x - 1 = 1 - y = z$.
4. $\tan^{-1}\dfrac{a}{k}$.
5. $\tan^{-1}\dfrac{t}{\sqrt{2}}$.
6. Normal $x - 1 = \dfrac{y-2}{2} = \dfrac{z-2}{2}$,
   tangent plane $x + 2y + 2z = 9$.
7. Normal $x - 3y = 6$, $z = 1$,
   tangent plane $3x + y - 8 = 0$.
8. Normal $\dfrac{x-3}{3} = \dfrac{y-4}{4} = \dfrac{5-z}{5}$,
   tangent plane $3x + 4y - 5z = 0$.
9. Normal $\dfrac{x-1}{1} = \dfrac{y-3}{3} = \dfrac{5-z}{1}$,
   tangent plane $x + 3y - z - 5 = 0$.
10. $y = z$, $x + y = \pm 2$.
11. The box should have a square base with side equal twice the depth.
12. $x = \frac{1}{3}(x_1 + x_2 + x_3)$, $y = \frac{1}{3}(y_1 + y_2 + y_3)$.
13. $(0, 0, \pm 1)$.
14. The altitude of the cone is twice the altitude of the cylinder.
15. The points P, Q, R, S lie in a horizontal plane, PQ, QR make equal angles with the mirror at Q, and QR, RS make equal angles with the mirror at R.

Pages 246, 247

1. $5\frac{1}{4}$.
2. $\frac{1}{3}$.
3. $\frac{1}{6}\ln 2$.
4. $\frac{1}{4}\pi a^2$.
5. $\dfrac{\pi}{k}$.
6. $\frac{1}{4}\pi a^2$.
7. $\frac{8}{3}$.
8. $\frac{10}{3}a^2$.
9. π.
10. $\frac{8}{3}$.
11. 2π.
12. $\frac{1}{24}$.
13. $\frac{8}{3}a^4$.
14. 4.
15. $\frac{2}{3}Ma^2$.
16. $\frac{1}{2}\pi a^5$.
17. $(\frac{8}{5}, \frac{7}{5})$.
18. $(\frac{12}{5}a, -2a)$.
19. $(\frac{3}{2}, \frac{3}{4}, \frac{1}{2})$.
20. $\frac{208}{15}\pi a^3$.

Pages 251, 252

1. $\frac{1}{8}\pi a^4$.
2. $\frac{1}{2}\pi a^2$.
3. $\frac{1}{4}\pi$.
4. $\frac{1}{6}\pi a^3$.
5. $(a + \frac{1}{2}\Delta a)\Delta a\, \Delta\alpha$.
6. $\dfrac{\pi}{2} \pm \frac{2}{3}$.
7. $\frac{1}{4} a^4(\alpha - \sin\alpha\cos\alpha)$.
8. On the bisector at distance $\dfrac{2\,a\sin\alpha}{3\,\alpha}$ from the center.
9. $\frac{45}{2}\pi a^4$.
10. $\frac{4}{3}\pi a^3(8 - 3\sqrt{3})$.
11. $\frac{3}{10} Ma^2$.
12. $\frac{2}{5} Ma^2$.
13. $\frac{3}{2} Ma^2$.
14. $\frac{8}{3}\pi a^3$.
15. $\frac{32}{9} a^3$.
16. $\frac{1}{2}\pi a^3$.

Pages 255, 256

1. $3\sqrt{14}$.
2. $16\,\pi$.
3. Two areas, each equal to $\pi a^2\sqrt{2}$.
4. $\pi a^2\sqrt{3}$.
5. 4.
6. $8\,a^2$.
7. $\frac{2}{3}\pi\,(3\sqrt{3} - 1)\,a^2$.
8. $2\,\pi\sqrt{6}$.
9. $\pi\sqrt{15}$
10. $8\,\pi a^2\,(2 - \sqrt{3})$.

Page 261

1. $\frac{1}{6}$.
2. $\frac{1}{30}$.
4. $\frac{4}{15}$.
5. $\dfrac{\pi}{2}$.
6. $\frac{32}{9} a^2h$.
7. $\frac{3}{8}\pi a^3$.
8. $\dfrac{\pi}{2}$.
9. $\frac{16}{45} a^5$.
10. $(\frac{3}{8}a,\ \frac{3}{8}b,\ \frac{3}{8}c)$.
11. $(0,\ 0,\ \frac{2}{3}c)$.
12. $\frac{4}{15}\pi abc\,(a^2 + b^2)$.

Pages 267, 268

1. $\frac{19}{6}\pi$.
2. Two regions each of volume $\frac{1}{3}\pi a^3\,(2 - \sqrt{2})$.
3. $\frac{32}{9} a^3$.
4. π.
5. $\frac{3}{4}\pi a^3$.
6. $\frac{3}{20} M(a^2 + 4\,h^2)$.
7. $\frac{7}{5} Ma^2$.
8. On the axis four-fifths of the distance from the vertex to the base.
9. On the axis at distance $\frac{3}{8}a(1 + \cos\alpha)$ from the vertex.
10. At distance $\frac{9}{16}a$ from the center.
11. $\frac{67}{480}\pi a^5$.
12. $\frac{241}{1920}\pi a^5$.
13. $\frac{8}{15}\pi\rho(b^5 - a^5)$.
14. 1045.4.
15. 627.8
16. 4128.5.

17. On the radius perpendicular to the plane face of the hemisphere at distance $\frac{3}{8}\frac{b^4 - a^4}{b^3 - a^3}$ from the center.

18. $M(b^2 + \frac{3}{4} a^2)$.

Pages 275, 276

13. 0.9397.
14. 0.1763.
15. 1.0515.
16. 0.5878.
17. 1.6487.
18. .1823.

Pages 283, 284

1. $$\sin x = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right) - \tfrac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \tfrac{1}{12}\sqrt{3}\left(x - \frac{\pi}{6}\right)^3 + \cdots.$$

2. $$\cos x = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}\left(x - \frac{\pi}{3}\right) - \tfrac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \tfrac{1}{12}\sqrt{3}\left(x - \frac{\pi}{3}\right)^3 + \cdots.$$

3. $$\sqrt{9 + x^2} = 5 + \tfrac{4}{5}(x - 4) + \tfrac{9}{250}(x - 4)^2 - \tfrac{18}{3125}(x - 4)^3 + \cdots.$$

4. $$\tan^{-1} x = \frac{\pi}{4} + \tfrac{1}{2}(x - 1) - \tfrac{1}{4}(x - 1)^2 + \tfrac{1}{12}(x - 1)^3 + \cdots.$$

5. 0.9063.
6. 0.2588.
7. 1.4281.
8. 1.7918.
10. 1.5557.
11. 2.9625.

Pages 290, 291

1. Approximate value 2.379.
2. 0.7593.
3. 8.533.
8. 0.7854.
9. 28.70.
10. 0.2463.
11. 3.190.
12. 21.48.
13. 31.03.
14. 0.5116.
15. 3.059.
16. $\lambda - \frac{\lambda^3}{3(3!)} + \frac{\lambda^5}{5(5!)} - \frac{\lambda^7}{7(7!)} + \cdots$
17. 2.937.

Page 297

5. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0.$
6. $x\,dy - y(x + 1)\,dx = 0.$
7. $\frac{d^2y}{dx^2} + y = 0.$
8. $x^3 \frac{d^3y}{dx^3} + 6x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} - 4y = 0.$
9. $y\,dx = x\,dy.$

Pages 307–309

1. $x^2 - y^2 = cx^2y^2.$
2. $\tan^2 x - \cot^2 y = c.$
3. $y^2 + 1 = c\,(x^2 - 1).$
4. $x^2y^2 + x^2 - y^2 = c.$
5. $x^3 + x^2y - xy^2 - y^3 = c.$
6. $y^2 = cx^2\,(y^2 + 1).$
7. $x^2 + y^2 = ce^{2ay}.$
8. $xy = c\,(y - 1).$
9. $y = ce^{ax} + \dfrac{1}{b - a}\,e^{bx}.$
10. $y = cx^2 - \dfrac{1}{x}.$
11. $y = cx^2e^{-\frac{3}{x}}.$
12. $x^2y = x + cy.$
13. $y = (1 - x^2)\,(x + c).$
14. $y = c\sin x - a.$
15. $7\,x^3 = y\,(x^7 + c).$
16. $\dfrac{1}{y^2} = x + \dfrac{1}{2} + ce^{2x}.$
17. $x^4 + 4\,y\,(x^2 - 1)^{\frac{3}{2}} = c.$
18. $\ln\,(x^2 + xy + y^2) + \dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{x + 2\,y}{x\sqrt{3}} = c.$
19. $x^2 - y^2 = cx.$
20. $y^2 + 2\,xy = c.$
21. $x^4 - 4\,x^3y + y^4 = c.$
22. $\dfrac{x}{y^2} = c - e^{-y}.$
23. $e^{\frac{x}{y}} + \ln x = c.$
24. $x + 2y + \ln\,(x + y - 2) = c.$
25. $y^3 = ce^x - x - 1.$
26. $e^y = \frac{1}{2}\,e^x + ce^{-x}.$
27. $y = \dfrac{c}{2} - \dfrac{x^2}{2\,c}.$
28. $y = \frac{1}{2}\,x^2 + c$, or $y = ce^x.$
29. $y^2 = 2\,cx + c^2.$
30. $q = Ec\left(1 - e^{-\frac{t}{Rc}}\right).$
31. $i = Ie^{-\frac{R}{L}t} + \dfrac{E}{R^2 + \alpha^2L^2}\left[R\sin\alpha t - L\alpha\left(\cos\alpha t - e^{-\frac{R}{L}t}\right)\right].$
32. $y^3 = 8\,e^{x-2}.$
33. $y^2 = 2\,ax.$
34. $y = cx^2.$
35. $x = a\ln\dfrac{y}{a + \sqrt{a^2 - y^2}} + \sqrt{a^2 - y^2} + c.$
36. $y^2 + (x - c)^2 = a^2.$
37. $y = \dfrac{c}{2}\,e^{\frac{x}{a}} + \dfrac{a^2}{2\,c}\,e^{-\frac{x}{a}}.$
38. $r = e^{\theta}.$
39. $r = c\sin\theta.$
40. $r = a\sec\,(\theta + c).$
41. $a\theta = \sqrt{r^2 - a^2} - a\sec^{-1}\dfrac{r}{a} + c.$
42. $y = e^{\frac{x}{k}}.$
43. A circle.
44. A straight line.
45. A circle with the fixed point on its circumference or at its center.
46. The logarithmic spirals $r = ce^{k\theta}.$
47. 99.9964.

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1. $y = c_1 \ln x - \frac{1}{4} x^2 + c_2.$
2. $y = x + c_1 x e^x + c_2.$
3. $y = c_1 e^{ax} + c_2 e^{-ax}.$
4. $y = c_1 \sin ax + c_2 \cos ax.$
5. $t = \int \sqrt{\dfrac{s}{2k + c_1 s}}\, ds + c_2.$
6. $s = \dfrac{1}{a^2} \ln (c_1 e^{abt} - e^{-abt}) + c_2.$
7. $y = \dfrac{c_1}{4} x^2 - \dfrac{1}{2c_1} \ln x + c_2.$
8. $y = \dfrac{1}{2c_1} [e^{c_1 x + c_2} + e^{-(c_1 x + c_2)}].$
9. $y = c_1 + c_2 e^{4x}.$
10. $y = c_1 e^{6x} + c_2 e^{-x}.$
11. $y = (c_1 + c_2 x) e^{3x}.$
12. $y = c_1 \cos x + c_2 \sin x.$
13. $y = c_1 + c_2 e^{-x} + c_3 e^{3x}.$
14. $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x.$
15. $y = e^x [c_1 \cos (x\sqrt{2}) + c_2 \sin (x\sqrt{2})].$
16. $y = e^{-\frac{1}{2}x} \left[c_1 \cos \dfrac{x\sqrt{3}}{2} + c_2 \sin \dfrac{x\sqrt{3}}{2} \right].$
17. $y = c_1 e^x + c_2 e^{-x} + c_3 e^{x\sqrt{2}} + c_4 e^{-x\sqrt{2}}.$
18. $y = (c_1 + c_2 x + c_3 x^2) e^x.$
19. $y = x + 3 + c_1 \cos x + c_2 \sin x.$
20. $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} e^x.$
21. $y = c_1 e^{2x} + c_2 e^{-3x} - \frac{1}{6} x^2 - \frac{1}{18} x - \frac{7}{108}.$
22. $y = ce^x - \frac{1}{2} (\sin x + \cos x).$
23. $y = c_1 + c_2 e^{2x} - \frac{1}{2} x^2 + x.$
24. $y = c_1 e^{-x} + c_2 e^{-5x} + \frac{1}{5} x - \frac{6}{25} + \frac{1}{32} e^{3x}.$
25. $y = c_1 e^{ax} + c_2 e^{-ax} + \dfrac{x}{2a} e^{ax}.$
26. $y = e^{\frac{1}{2}x} \left[c_1 \cos \dfrac{x\sqrt{3}}{2} + c_2 \sin \dfrac{x\sqrt{3}}{2} \right] - \dfrac{1}{13} (2 \sin 2x + 3 \cos 2x).$
27. $y = c_1 e^x + e^{-\frac{1}{2}x} \left[c_1 \cos \dfrac{x\sqrt{3}}{2} + c_2 \sin \dfrac{x\sqrt{3}}{2} \right] - x^3 + x^2 - 6.$
28. $y = c_1 e^x + c_2 e^{3x} - \frac{1}{2} e^{2x} \sin x.$
29. $y = c_1 e^{3x} + c_2 e^{-3x} + \frac{1}{37} e^{3x} (6 \sin x - \cos x).$
30. $y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + \frac{1}{1088} (4 \cos 4x - \sin 4x).$
31. $y = c_1 \cos 2x + (c_2 + \frac{1}{4} x) \sin 2x.$

32. $y = e^{-x}(c_1 + c_2x + \frac{1}{2}x^2) + \frac{1}{4}e^x$.
33. $x = c_1 \cos t + c_2 \sin t + \frac{1}{2}(e^t - e^{-t})$,
$y = c_2 \cos t - c_1 \sin t + \frac{1}{2}(e^t - e^{-t})$.
34. $y = c_1 \cos t + c_2 \sin t - 1$,
$x = (c_1 + c_2) \cos t + (c_2 - c_1) \sin t - 3$.
35. $x = c_1e^t + c_2e^{-3t}$,
$y = -c_1e^t + 3\,c_2e^{-3t} + \cos t$.
36. $x = c_1e^t + c_2e^{-t} + c_3 \cos t + c_4 \sin t$,
$y = c_1e^t + c_2e^{-t} - c_3 \cos t - c_4 \sin t$.
37. $y = x$.
38. $2\,y^{\frac{1}{2}} = x + 2$.
39. $s = \frac{g}{k}t + \frac{g}{k^2}(e^{-kt} - 1)$.
40. $s = \frac{1}{k}\ln\left(\frac{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}{2}\right)$.
41. $s = b \cos (kt)$.
42. About 7 miles per second.
43. About $42\frac{1}{2}$ minutes.
44. $t = \sqrt{\frac{5}{g}}\ln(5 + \sqrt{24})$.
45. $t = \frac{3}{\sqrt{g}}\ln(9 + 4\sqrt{5})$.

TABLE OF INTEGRALS

1. $\int u^n\,du = \dfrac{u^{n+1}}{n+1}$, if n is not -1.

2. $\int \dfrac{du}{u} = \ln u$.

3. $\int \dfrac{du}{u^2+a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a}$.

4. $\int \dfrac{du}{u^2-a^2} = \dfrac{1}{2a}\ln\dfrac{u-a}{u+a}$.

5. $\int e^u\,du = e^u$.

6. $\int a^u\,du = \dfrac{a^u}{\ln a}$.

Integrals of Trigonometric Functions

7. $\int \sin u\,du = -\cos u$.

8. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u = \frac{1}{2}(u - \sin u\cos u)$.

9. $\int \sin^4 u\,du = \frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u$.

10. $\int \sin^6 u\,du = \frac{5}{16}u - \frac{1}{4}\sin 2u + \frac{1}{48}\sin^3 2u + \frac{3}{64}\sin 4u$.

11. $\int \cos u\,du = \sin u$.

12. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u = \frac{1}{2}(u + \sin u\cos u)$.

13. $\int \cos^4 u\,du = \frac{3}{8}u + \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u$.

14. $\int \cos^6 u\,du = \frac{5}{16}u + \frac{1}{4}\sin 2u - \frac{1}{48}\sin^3 2u + \frac{3}{64}\sin 4u$.

15. $\int \tan u\,du = -\ln\cos u$.

16. $\int \cot u\,du = \ln\sin u$.

17. $\int \sec u\, du = \ln(\sec u + \tan u) = \ln \tan\left(\frac{u}{2} + \frac{\pi}{4}\right).$

18. $\int \sec^2 u\, du = \tan u.$

19. $\int \sec^3 u\, du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln(\sec u + \tan u).$

20 $\int \csc u\, du = \ln(\csc u - \cot u) = \ln \tan \frac{u}{2}.$

21. $\int \csc^2 u\, du = -\cot u.$

22. $\int \csc^3 u\, du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln(\csc u - \cot u).$

Integrals containing $\sqrt{a^2 - u^2}$

23. $\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}.$

24. $\int u^2\sqrt{a^2 - u^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{a^2 - u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a}.$

25. $\int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} + a\ln\frac{a - \sqrt{a^2 - u^2}}{u}.$

26. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a}.$

27. $\int \frac{u^2\, du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}.$

28. $\int \frac{du}{u\sqrt{a^2 - u^2}} = \frac{1}{a}\ln\frac{a - \sqrt{a^2 - u^2}}{u}.$

29. $\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u}.$

30. $\int (a^2 - u^2)^{\frac{3}{2}}\, du = \frac{u}{8}(5a^2 - 2u^2)\sqrt{a^2 - u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a}.$

31. $\int \frac{du}{(a^2 - u^2)^{\frac{3}{2}}} = \frac{u}{a^2\sqrt{a^2 - u^2}}.$

Integrals containing $\sqrt{u^2 - a^2}$

32. $\int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left(u + \sqrt{u^2 - a^2}\right).$

33. $\int u^2\sqrt{u^2 - a^2}\, du = \frac{u}{8}(2u^2 - a^2)\sqrt{u^2 - a^2} - \frac{a^4}{8}\ln\left(u + \sqrt{u^2 - a^2}\right).$

34. $$\int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a \sec^{-1} \frac{u}{a}.$$

35. $$\int \frac{du}{\sqrt{u^2 - a^2}}\, du = \ln \left(u + \sqrt{u^2 - a^2}\right).$$

36. $$\int \frac{u^2\, du}{\sqrt{u^2 - a^2}} = \frac{u}{2} \sqrt{u^2 - a^2} + \frac{a^2}{2} \ln \left(u + \sqrt{u^2 - a^2}\right).$$

37. $$\int \frac{du}{u \sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a}.$$

38. $$\int \frac{du}{u^2 \sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u}.$$

39. $$\int (u^2 - a^2)^{\frac{3}{2}}\, du = \frac{u}{8}(2u^2 - 5a^2) \sqrt{u^2 - a^2} + \frac{3a^4}{8} \ln \left(u + \sqrt{u^2 - a^2}\right).$$

40. $$\int \frac{du}{(u^2 - a^2)^{\frac{3}{2}}} = -\frac{u}{\sqrt{u^2 - a^2}}.$$

Integrals containing $\sqrt{u^2 + a^2}$

41. $$\int \sqrt{u^2 + a^2}\, du = \frac{u}{2} \sqrt{u^2 + a^2} + \frac{a^2}{2} \ln \left(u + \sqrt{u^2 + a^2}\right).$$

42. $$\int u^2 \sqrt{u^2 + a^2}\, du = \frac{u}{8}(2u^2 + a^2) \sqrt{u^2 + a^2} - \frac{a^4}{8} \ln \left(u + \sqrt{u^2 + a^2}\right).$$

43. $$\int \frac{\sqrt{u^2 + a^2}}{u}\, du = \sqrt{u^2 + a^2} + a \ln \frac{\sqrt{u^2 + a^2} - a}{u}.$$

44. $$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln \left(u + \sqrt{u^2 + a^2}\right).$$

45. $$\int \frac{u^2\, du}{\sqrt{u^2 + a^2}} = \frac{u}{2} \sqrt{u^2 + a^2} - \frac{a^2}{2} \ln \left(u + \sqrt{u^2 + a^2}\right).$$

46. $$\int \frac{du}{u \sqrt{u^2 + a^2}} = \frac{1}{a} \ln \frac{\sqrt{u^2 + a^2} - a}{u}.$$

47. $$\int \frac{du}{u^2 \sqrt{u^2 + a^2}} = -\frac{\sqrt{u^2 + a^2}}{a^2 u}.$$

48. $$\int (u^2 + a^2)^{\frac{3}{2}}\, du = \frac{u}{8}(2u^2 + 5a^2) \sqrt{u^2 + a^2} + \frac{3a^4}{8} \ln \left(u + \sqrt{u^2 + a^2}\right).$$

49. $$\int \frac{du}{(u^2 + a^2)^{\frac{3}{2}}} = \frac{u}{a^2 \sqrt{u^2 + a^2}}.$$

Other Integrals

50. $$\int \sqrt{\frac{px+q}{ax+b}}\,dx$$
$$= \frac{1}{a}\sqrt{(ax+b)(px+q)} - \frac{bp-aq}{a\sqrt{ap}} \ln\left(\sqrt{p(ax+b)} + \sqrt{a(px+q)}\right)$$
$$= \frac{1}{a}\sqrt{(ax+b)(px+q)} - \frac{bp-aq}{a\sqrt{-ap}} \tan^{-1} \frac{\sqrt{-ap(ax+b)}}{a\sqrt{px+q}}.$$

51. $$\int e^{ax} \sin bx\,dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2}.$$

52. $$\int e^{ax} \cos bx\,dx = \frac{e^{ax}(b \sin bx + a \cos bx)}{a^2+b^2}.$$

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N	0	1	2	3	4	5	6	7	8	9
0	. .	0 0000	0 6931	1 0986	1 3863	1 6094	1 7918	1 9459	2 0794	2 1972
1	2 3026	2 3979	2 4849	2 5649	2 6391	2 7081	2 7726	2 8332	2 8904	2 9444
2	9957	3 0445	3 0910	3 1355	3 1781	3 2189	3 2581	3 2958	3 3322	3 3673
3	3 4012	4340	4657	4965	5264	5553	5835	6109	6376	6636
4	6889	7136	7377	7612	7842	8067	8286	8501	8712	8918
5	9120	9318	9512	9703	9890	4 0073	4 0254	4 0431	4 0604	4 0775
6	4 0943	4 1109	4 1271	4 1431	4 1589	1744	1897	2047	2195	2341
7	2485	2627	2767	2905	3041	3175	3307	3438	3567	3694
8	3820	3944	4067	4188	4308	4427	4543	4659	4773	4886
9	4998	5109	5218	5326	5433	5539	5643	5747	5850	5951
10	6052	6151	6250	6347	6444	6540	6634	6728	6821	6913
11	7005	7095	7185	7274	7362	7449	7536	7622	7707	7791
12	7875	7958	8040	8122	8203	8283	8363	8442	8520	8598
13	8675	8752	8828	8903	8978	9053	9127	9200	9273	9345
14	9416	9488	9558	9628	9698	9767	9836	9904	9972	5 0039
15	5.0106	5 0173	5 0239	5 0304	5 0370	5.0434	5 0499	5 0562	5 0626	0689
16	0752	0814	0876	0938	0999	1059	1120	1180	1240	1299
17	1358	1417	1475	1533	1591	1648	1705	1761	1818	1874
18	1930	1985	2040	2095	2149	2204	2257	2311	2364	2417
19	2470	2523	2575	2627	2679	2730	2781	2832	2883	2933
20	2983	3033	3083	3132	3181	3230	3279	3327	3375	3423
21	3471	3519	3566	3613	3660	3706	3753	3799	3845	3891
22	3936	3982	4027	4072	4116	4161	4205	4250	4293	4337
23	4381	4424	4467	4510	4553	4596	4638	4681	4723	4765
24	4806	4848	4889	4931	4972	5013	5053	5094	5134	5175
25	5215	5255	5294	5334	5373	5413	5452	5491	5530	5568
26	5607	5645	5683	5722	5759	5797	5835	5872	5910	5947
27	5984	6021	6058	6095	6131	6168	6204	6240	6276	6312
28	6348	6384	6419	6454	6490	6525	6560	6595	6630	6664
29	6699	6733	6768	6802	6836	6870	6904	6937	6971	7004
30	7038	7071	7104	7137	7170	7203	7236	7268	7301	7333
31	7366	7398	7430	7462	7494	7526	7557	7589	7621	7652
32	7683	7714	7746	7777	7807	7838	7869	7900	7930	7961
33	7991	8021	8051	8081	8111	8141	8171	8201	8230	8260
34	8289	8319	8348	8377	8406	8435	8464	8493	8522	8551
35	8579	8608	8636	8665	8693	8721	8749	8777	8805	8833
36	8861	8889	8916	8944	8972	8999	9026	9054	9081	9108
37	9135	9162	9189	9216	9243	9269	9296	9322	9349	9375
38	9402	9428	9454	9480	9506	9532	9558	9584	9610	9636
39	9661	9687	9713	9738	9764	9789	9814	9839	9865	9890
40	9915	9940	9965	9989	6 0014	6.0039	6 0064	6 0088	6 0113	6 0137
41	6 0162	6 0186	6 0210	6 0234	0259	0283	0307	0331	0355	0379
42	0403	0426	0450	0474	0497	0521	0544	0568	0591	0615
43	0638	0661	0684	0707	0730	0753	0776	0799	0822	0845
44	0868	0890	0913	0936	0958	0981	1003	1026	1048	1070
45	1092	1115	1137	1159	1181	1203	1225	1247	1269	1291
46	1312	1334	1356	1377	1399	1420	1442	1463	1485	1506
47	1527	1549	1570	1591	1612	1633	1654	1675	1696	1717
48	1738	1759	1779	1800	1821	1841	1862	1883	1903	1924
49	1944	1964	1985	2005	2025	2046	2066	2086	2106	2126
50	2146	2166	2186	2206	2226	2246	2265	2285	2305	2324
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
50	6 2146	6 2166	6 2186	6 2206	6 2226	6 2246	6 2265	6 2285	6 2305	6 2324
51	2344	2364	2383	2403	2422	2442	2461	2480	2500	2519
52	2538	2558	2577	2596	2615	2634	2653	2672	2691	2710
53	2729	2748	2766	2785	2804	2823	2841	2860	2879	2897
54	2916	2934	2953	2971	2989	3008	3026	3044	3063	3081
55	3099	3117	3135	3154	3172	3190	3208	3226	3244	3261
56	3279	3297	3315	3333	3351	3368	3386	3404	3421	3439
57	3456	3474	3491	3509	3526	3544	3561	3578	3596	3613
58	3630	3648	3665	3682	3699	3716	3733	3750	3767	3784
59	3801	3818	3835	3852	3869	3886	3902	3919	3936	3953
60	3969	3986	4003	4019	4036	4052	4069	4085	4102	4118
61	4135	4151	4167	4184	4200	4216	4232	4249	4265	4281
62	4297	4313	4329	4345	4362	4378	4394	4409	4425	4441
63	4457	4473	4489	4505	4520	4536	4552	4568	4583	4599
64	4615	4630	4646	4661	4677	4693	4708	4723	4739	4754
65	4770	4785	4800	4816	4831	4846	4862	4877	4892	4907
66	4922	4938	4953	4968	4983	4998	5013	5028	5043	5058
67	5073	5088	5103	5117	5132	5147	5162	5177	5191	5206
68	5221	5236	5250	5265	5280	5294	5309	5323	5338	5352
69	5367	5381	5396	5410	5425	5439	5453	5468	5482	5497
70	5511	5525	5539	5554	5568	5582	5596	5610	5624	5639
71	5653	5667	5681	5695	5709	5723	5737	5751	5765	5779
72	5793	5806	5820	5834	5848	5862	5876	5889	5903	5917
73	5930	5944	5958	5971	5985	5999	6012	6026	6039	6053
74	6067	6080	6093	6107	6120	6134	6147	6161	6174	6187
75	6201	6214	6227	6241	6254	6267	6280	6294	6307	6320
76	6333	6346	6359	6373	6386	6399	6412	6425	6438	6451
77	6464	6477	6490	6503	6516	6529	6542	6554	6567	6580
78	6593	6606	6619	6631	6644	6657	6670	6682	6695	6708
79	6720	6733	6746	6758	6771	6783	6796	6809	6821	6834
80	6846	6859	6871	6884	6896	6908	6921	6933	6946	6958
81	6970	6983	6995	7007	7020	7032	7044	7056	7069	7081
82	7093	7105	7117	7130	7142	7154	7166	7178	7190	7202
83	7214	7226	7238	7250	7262	7274	7286	7298	7310	7322
84	7334	7346	7358	7370	7382	7393	7405	7417	7429	7441
85	7452	7464	7476	7488	7499	7511	7523	7534	7546	7558
86	7569	7581	7593	7604	7616	7627	7639	7650	7662	7673
87	7685	7696	7708	7719	7731	7742	7754	7765	7776	7788
88	7799	7811	7822	7833	7845	7856	7867	7878	7890	7901
89	7912	7923	7935	7946	7957	7968	7979	7991	8002	8013
90	8024	8035	8046	8057	8068	8079	8090	8101	8112	8123
91	8134	8145	8156	8167	8178	8189	8200	8211	8222	8233
92	8244	8255	8265	8276	8287	8298	8309	8320	8330	8341
93	8352	8363	8373	8384	8395	8405	8416	8427	8437	8448
94	8459	8469	8480	8491	8501	8512	8522	8533	8544	8554
95	8565	8575	8586	8596	8607	8617	8628	8638	8648	8659
96	8669	8680	8690	8701	8711	8721	8732	8742	8752	8763
97	8773	8783	8794	8804	8814	8824	8835	8845	8855	8865
98	8876	8886	8896	8906	8916	8926	8937	8947	8957	8967
99	8977	8987	8997	9007	9017	9027	9037	9048	9058	9068
100	9078	9088	9098	9108	9117	9127	9137	9147	9157	9167
N	0	1	2	3	4	5	6	7	8	9